An Information-Theoretic Metric of System Complexity with Application to Engineering System Design

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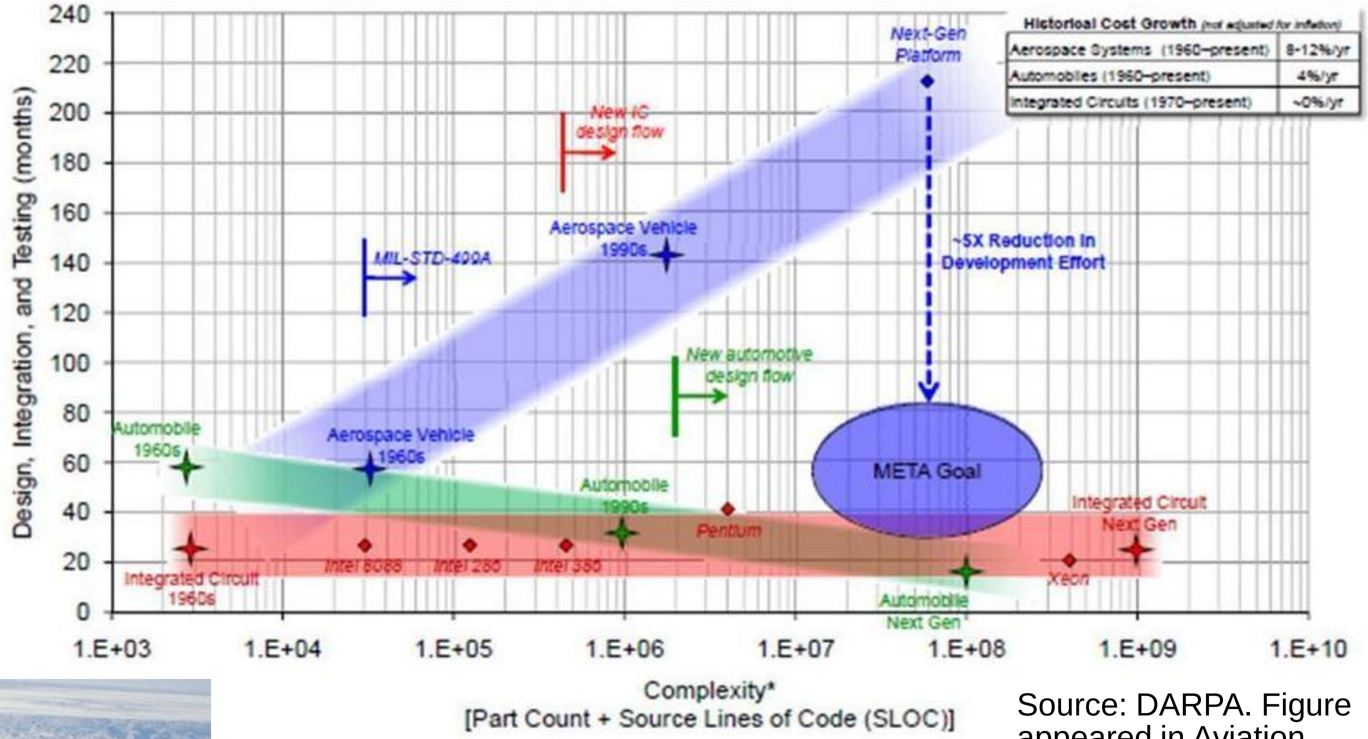
Motivation

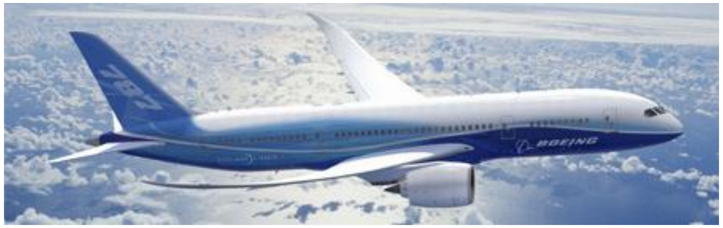
Development times and costs of aerospace systems have reached unsustainable levels – and are getting worse.



Historical Cost Growth *(not adjusted for inflation)*

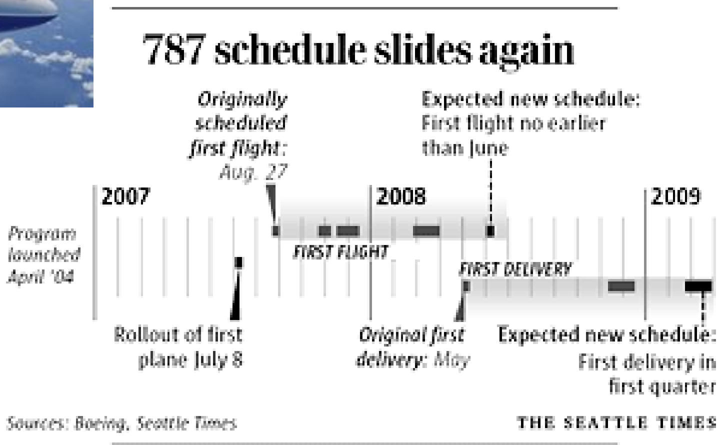
Aerospace Systems (1960–present)	8-12%/yr
Automobiles (1960–present)	4%/yr
Integrated Circuits (1970–present)	~0%/yr

Source: DARPA. Figure appeared in Aviation Week & Space Tech., Nov. 1-8 2011.



Source: www.boeing.com.

The Boeing 787 program has incurred significant cost and schedule overruns due to unexpected integration issues.



787 schedule slides again

Sources: Boeing, Seattle Times

THE SEATTLE TIMES

Outline

- Background
- Definition of system complexity
- Total uncertainty quantification for computer models
- Derivation of sensitivity indices
- Demonstration problems
- Conclusions

Background – Complexity

- Complexity in system design is an elusive concept…
  - Qualitatively:
    - Nebulous middle ground between order and chaos (Weaver 1948)
    - "I know it when I see it" (Johnson 1997)
  - Quantitatively:
    - Structure-based: source lines of code, number of parts, etc. (Griffin 1997)
    - Process-based: algorithmic complexity, computational complexity, etc. (Kolmogorov 1965, Chaitin 1969)
    - Information-based: information entropy, thermodynamic depth (Lloyd 1988)

Many metrics. The usefulness of each depends on context.

Background – Our context

- Generally agreed upon properties of a complex system
  - Consist of many parts
  - Parts interact
  - Difficult to model and understand
- Consider the design of a next generation infantry fighting vehicle
  - What are the quantities we truly care about when designing the vehicle?
    - Range
    - Acceleration
    - Quiet time duration
    - Armor capabilities
    - Cost
    - Development time
    - …

Source: www.inetres.com

Background – Information Entropy

Consider a random variable Y with probability mass function $p(y)$

The entropy of Y is defined as :

$$H(Y) = -\sum_i p(y_i)\log p(y_i),$$

where $y_1, y_2, \ldots$ are the values of y such that $p(y) \neq 0$

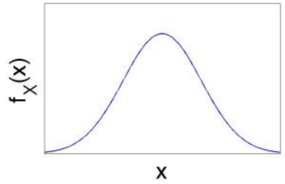
Consider a random variable X with probability density function $f_X(x)$

Differential entropy of X is defined as :

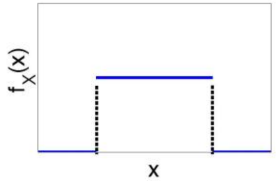
$$h(X) = -\int_{\mathbb{X}} f_X(x)\log f_X(x)dx$$

Examples :

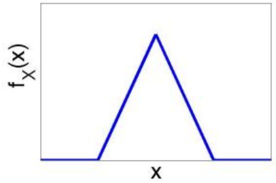
$$h(\mathcal{N}(\mu,\sigma^2)) = \frac{1}{2}\ln(2\pi e\sigma^2)$$



$$h(U[a,b]) = \ln(b-a)$$



$$h(T(a,b,c)) = \frac{1}{2} + \ln\left(\frac{b-a}{2}\right)$$



System Complexity

Proposed Definition: System Complexity

The potential of a system to exhibit unexpected behavior in the quantities of interest.

- Captures qualitative aspects of system complexity
  - notion of emergent behavior
  - Lack of understanding
- Can be quantitatively measured

Proposed Metric: System Complexity

Let $f_Q(q)$ be the probability density function of a quantity of interest. Then

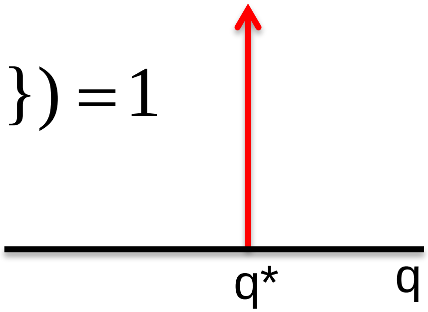
$$C(Q) \triangleq \exp\left\{-\int_{-\infty}^{\infty} f_Q(q)\ln f_Q(q)dq\right\} = \exp\{h(Q)\}$$

is a metric of system complexity as defined above.

Complexity Metric

- For the case where we have perfect knowledge, complexity $C(Q) = 0$

$$\Pr(\{Q = q^*\}) = 1$$



- For all other cases, $C(Q) \in [0, \infty)$
- For the case of a uniform random variable

$$Q \sim U[a, b]$$

$$C(Q) = \exp(\{h(Q)\} = \exp(\{\ln(b - a)\}) = b - a$$

Total UQ for Computer Models

- **Parametric uncertainty** – refers to uncertain inputs or parameters of a model
- **Parametric variability** – uncontrolled or unspecified conditions in inputs or parameters
- **Model discrepancy** – no model is perfect…
- **Code uncertainty** – uncertainty associated with not knowing the output of a computer model given any particular input configuration until the code is run

Model Discrepancy

- We <u>must</u> quantify model discrepancy
- From Kennedy and O'Hagan, 2001: "No model is perfect. Even if there is no parameter uncertainty, so that we know the true values of all the inputs required to make a particular prediction of the process being modeled, the predicted value will not equal the true value of the process. The discrepancy is model inadequacy."

$$\boxed{\text{Model Fidelity} \propto (\text{Model Discrepancy})^{-1}}$$

$$\overset{\text{Quantity of interest}}{q} = \overset{\text{Simulator}}{g(\mathbf{x})} + \overset{\text{Model Discrepancy}}{\epsilon(\mathbf{x})}$$

Code Uncertainty

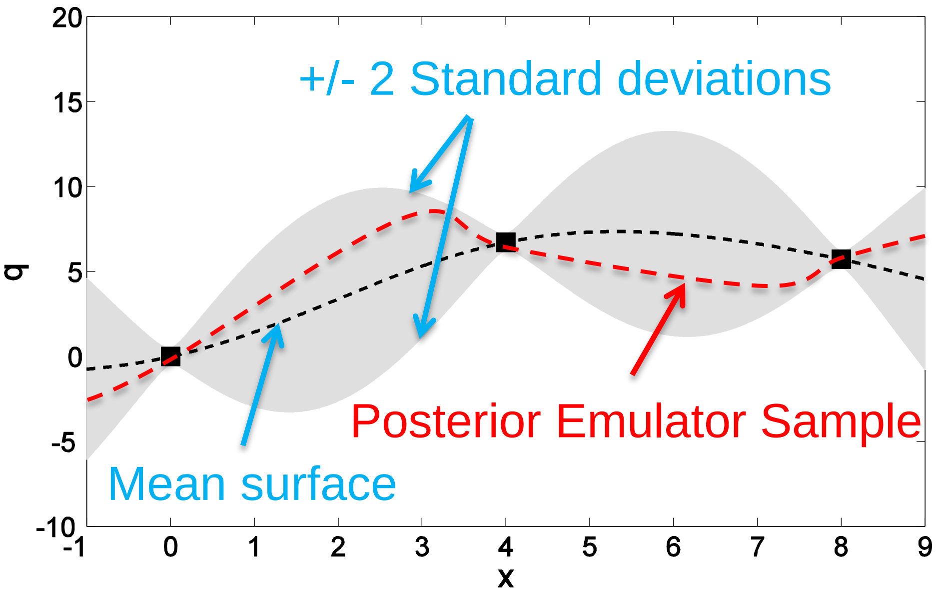
- Code uncertainty – uncertainty associated with not knowing the output of a computer model given any particular configuration until the code is run

Quantity of interest

Model Discrepancy

Emulator

$$q = \mathcal{G}(\mathbf{x}) + \epsilon(\mathbf{x})$$



<u>Gaussian Process Emulator</u>

$$\mathcal{G}(\mathbf{x}) \sim \mathcal{GP}(\mathrm{m}(\mathbf{x}), \mathrm{k}(\mathbf{x}, \mathbf{x}'))$$

$\mathrm{m}(\mathbf{x})$ is a mean function

$\mathrm{k}(\mathbf{x}, \mathbf{x}')$ is a covariance kernel

Complexity Metric Estimation

Must incorporate all sources of uncertainty

$f_Q(q)$ [Quantity of interest density]

$$h(Q^{\Delta}) = -\sum_{j=1}^{N} [\, f_Q(q^j)\Delta] \log[\, f_Q(q^j)\Delta + \log \Delta$$ [Entropy Estimate]

$$\hat{C}(Q) = \exp\{h(Q^{\Delta})\}$$ [Complexity estimate]

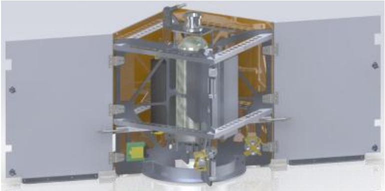
$$\hat{C}(Q \mid \mathcal{G} = G)$$ [Complexity estimate conditioned on emulator sample]

$$\tilde{C}(Q) = \max_{G}(\hat{C}(Q \mid \mathcal{G} = G))$$ [Complexity estimator]

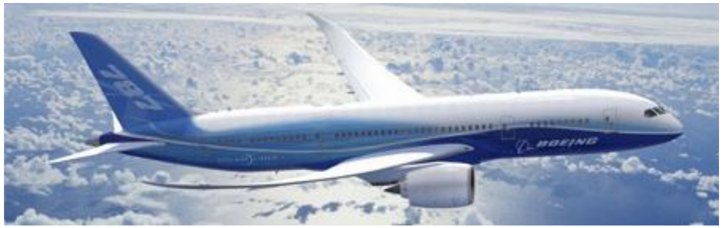
Identification of Key Contributors to Complexity

Complex System Design

Source: www.inetres.com

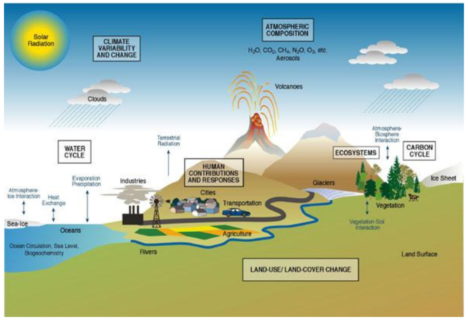


Cathode **A**node **S**pace **T**hruster for **O**rbital **R**epositioning Satellite



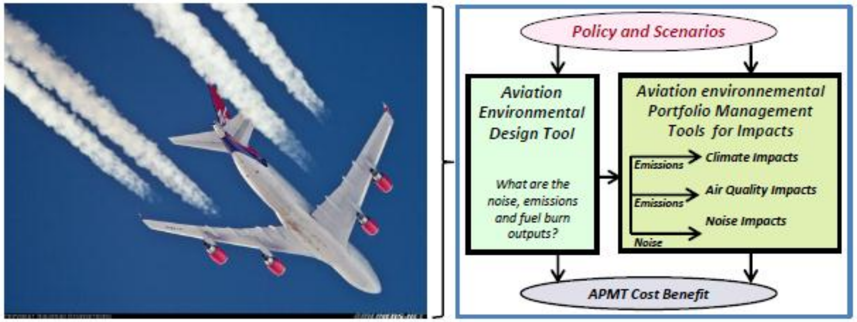
Source: www.boeing.com.

Complex System Analysis



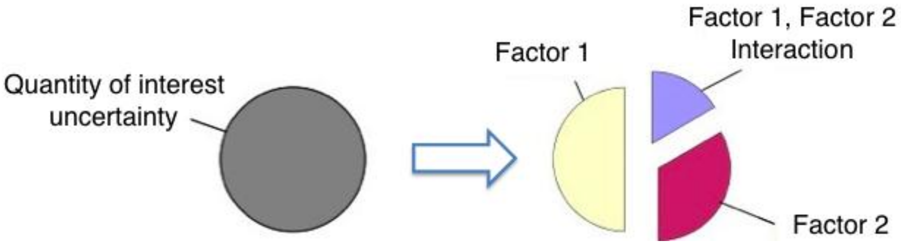
Source: blog.cunysustainablecities.org.

Global climate change



Source: www.airliners.net

Coupled Aviation-Environmental System



Variance-based approach (Homma 1996)

$$\operatorname{var}(Q) = \operatorname{var}(\mathbb{E}[Q \mid X_i]) + \mathbb{E}[\operatorname{var}(Q \mid X_i)]$$

$$S_i = \frac{\operatorname{var}(\mathbb{E}[Q \mid X_i])}{\operatorname{var}(Q)} = \frac{\operatorname{var}(Q) - \mathbb{E}[\operatorname{var}(Q \mid X_i)]}{\operatorname{var}(Q)}$$

Sensitivity Indices

For parametric uncertainty and variability

$$\hat{\eta}_i(\mathcal{G}) = \frac{\overbrace{\hat{C}(Q\,|\,\mathcal{G}=G)}^{\text{Initial complexity conditioned on emulator sample}} - \overbrace{\mathbb{E}_{X_i}[\hat{C}(Q\,|\,\mathcal{G}=G, X_i = x_i^j)]}^{\text{Expected complexity remaining once factor i is known}}}{\hat{C}(Q\,|\,\mathcal{G}=G)}$$

Average over the emulator samples to obtain sensitivity indices

$$\boxed{\tilde{\eta}_i = \mathbb{E}_{\mathcal{G}}[\hat{\eta}_i(\mathcal{G})]}$$

Sensitivity Indices

For model discrepancy

Recall:

$$\tilde{C}(Q) = \max_{G}(\hat{C}(Q \mid \mathcal{G} = G))$$

$$\tilde{\eta}_{\mathrm{MD}} = \frac{\overbrace{\tilde{C}(Q)}^{\text{Initial complexity}} - \overbrace{\tilde{C}(Q \mid \epsilon(\mathbf{x}) = 0)}^{\text{Complexity remaining if the model is assumed to be perfect}}}{\tilde{C}(Q)}$$

For code uncertainty

$$\tilde{\eta}_{\mathrm{CU}} = \frac{\overbrace{\tilde{C}(Q)}^{\text{Initial complexity}} - \overbrace{\mathbb{E}_{\mathcal{G}}[\hat{C}(Q \mid \mathcal{G} = G)]}^{\text{Expected complexity remaining if simulator was known at every point}}}{\tilde{C}(Q)}$$

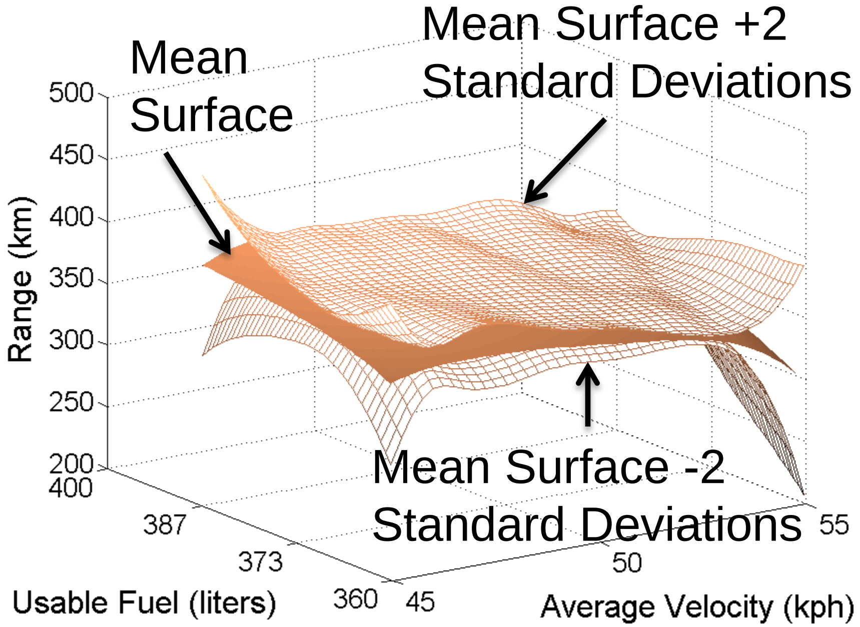
Demonstration 1

- Estimate the complexity of an IFV design with respect to **range** as the quantity of interest
- Parameters:
  - – Usable fuel ~ U[360,400] liters (parametric uncertainty)
  - – Average velocity ~ U[45,55] kph (parametric variability)
- Model discrepancy

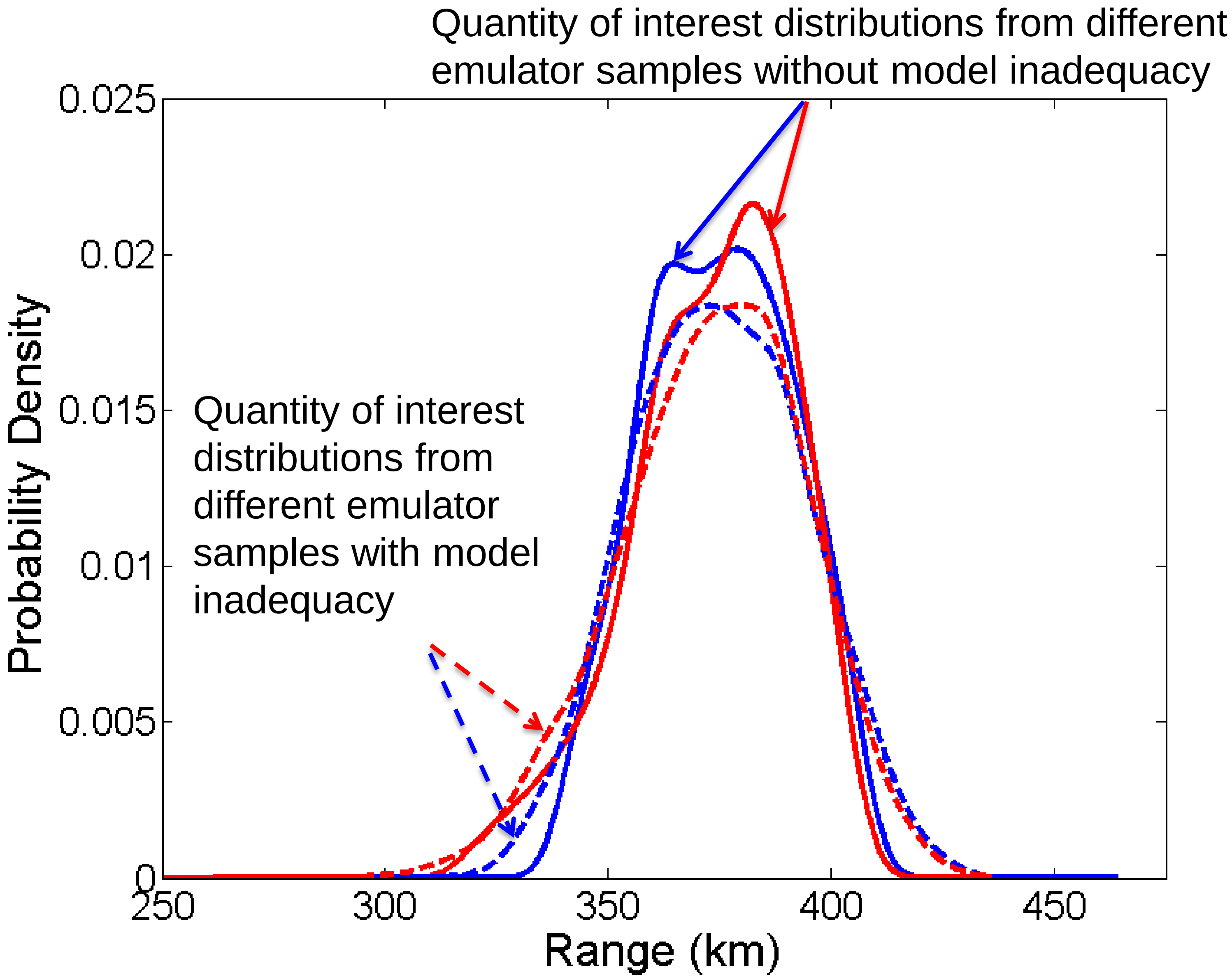
$$q = g(\mathbf{x}) + \epsilon(\mathbf{x})$$
$$\epsilon(\mathbf{x}) \sim N(0, 100\text{km}^2)$$

- Code uncertainty

Gaussian process emulator



Results – Quantity of Interest Densities



Quantity of interest distributions from different emulator samples without model inadequacy

Quantity of interest distributions from different emulator samples with model inadequacy

Results

Complexity	$\tilde{C}(Q) = 104\ \mathrm{km}$

Average velocity sensitivity	$\tilde{\eta}_{\mathrm{AV}} = 0.46$
Usable fuel sensitivity	$\tilde{\eta}_{\mathrm{F}} = 0.44$
Model discrepancy sensitivity	$\tilde{\eta}_{\mathrm{MD}} = 0.15$
Code uncertainty	$\tilde{\eta}_{\mathbf{CU}} = 0.16$

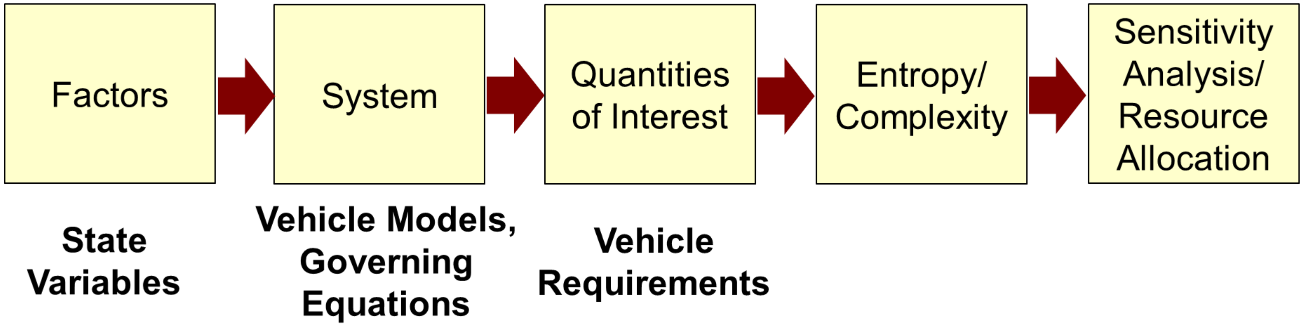
Allocate resources to learning more about the target velocity and fuel level

Demonstration 2

- Notional design process for a hybrid IFV
- Purpose is to demonstrate at a high level the role of sensitivity analysis and feedback
- Primarily the work of John Deyst and Chelsea He

Approach

- Specify the quantities of interest → **vehicle requirements**
- Specify the factors that influence the quantities of interest → **state variables**
- Decompose the vehicle design in terms of systems, subsystems, and components, and identify linking variables
- Use available vehicle design models to compute the quantities of interest
- Compute the probability of failure with respect to the quantities of interest
- Estimate complexity in the quantities of interest using Monte Carlo simulation
- Perform sensitivity analysis to identify sources of complexity
- Perform resource allocation to reduce complexity
- Iterate until feasible design is achieved that exhibits acceptable complexity in the quantities of interest



Factors

System

Quantities of Interest

Entropy/ Complexity

Sensitivity Analysis/ Resource Allocation

State Variables

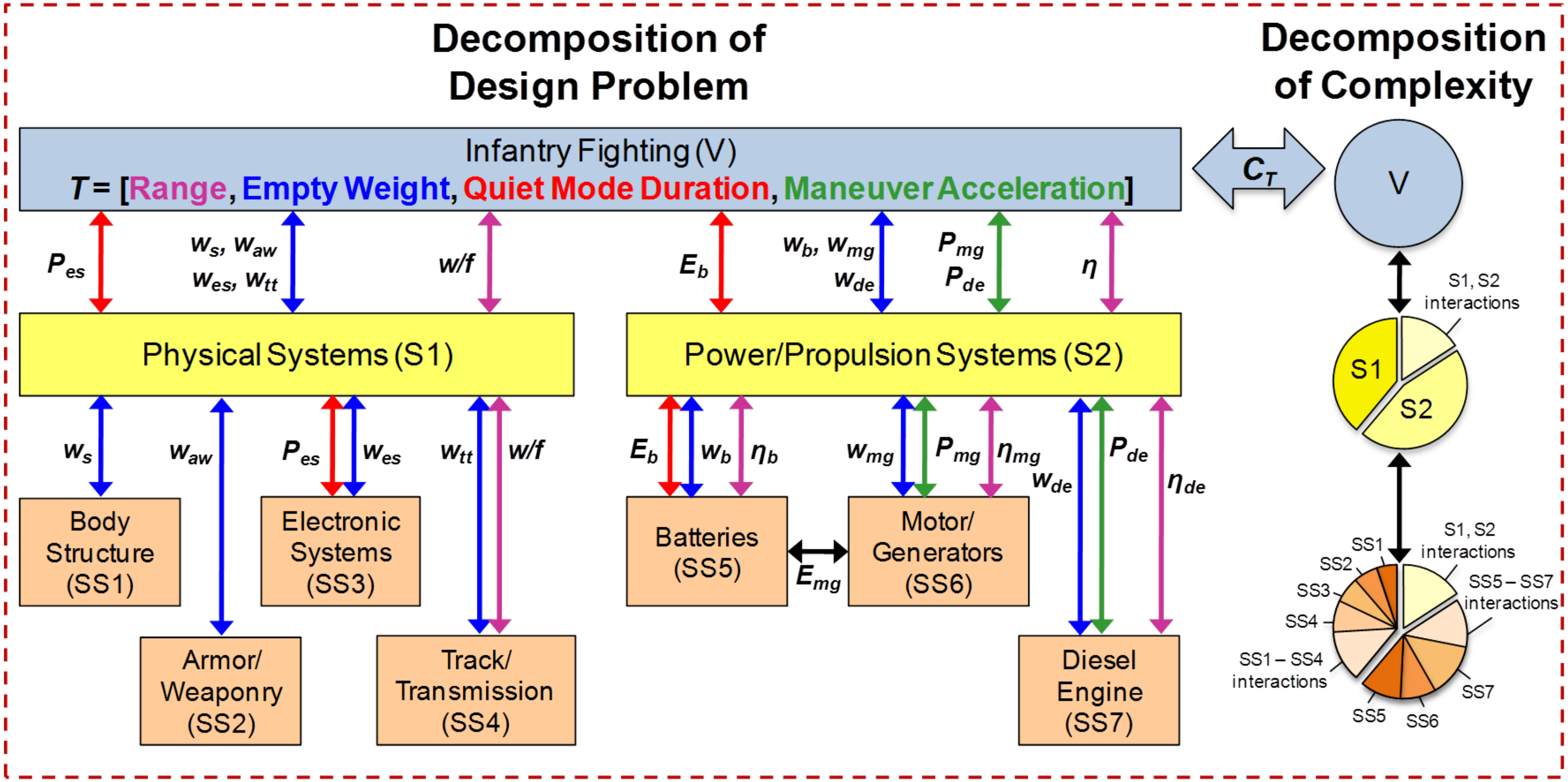
Vehicle Models, Governing Equations

Vehicle Requirements

Requirements and System Decomposition

Vehicle Requirements

- The hybrid IFV must have a range of at least 500 kilometers.
- The empty weight of the hybrid IFV must not exceed 25,000 kilograms.
- The hybrid IFV must operate in quiet mode for at least 8 hours.
- The hybrid IFV must achieve a maneuver acceleration of 0 to 10 m/s in 5 sec.



Iterate until feasible design achieved

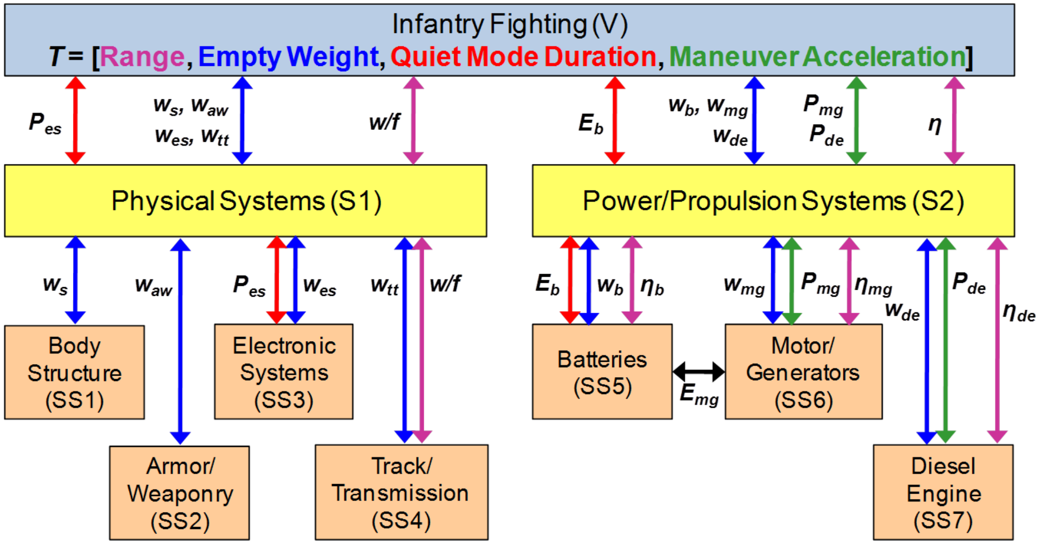
System Factors

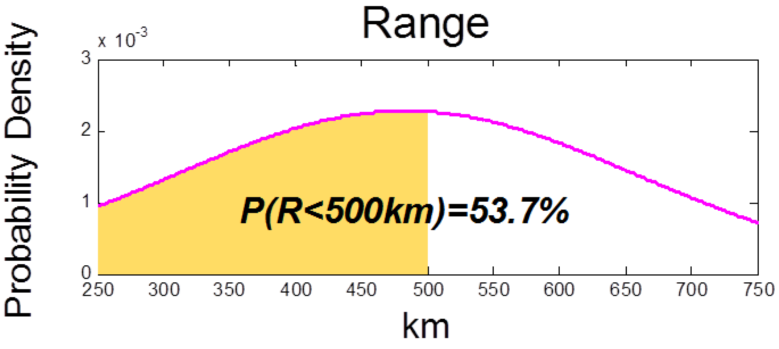
State Variable			Initial Value	Std. Dev.
1.	w/f	Weight to thrust ratio [–]	15	3
2.	η_{de}	Diesel engine efficiency [–]	0.3	0.06
3.	η_{b}	Battery charging efficiency [–]	0.9	0.18
4.	η_{mg}	Motor/generator efficiency [–]	0.8	0.16
5.	W_{fuel}	Fuel weight [kg]	400	80
6.	W_{p}	Payload weight [kg]	500	100
7.	W_{e}	Empty weight [kg]	24,000	4,800
8.	P_{es}	Quiet mode power [kW]	10	2
9.	E_{b}	Battery energy capacity [kWh]	80	16
10.	P_{de}	Diesel engine power [kW]	200	40
11.	P_{mg}	Motor/generator power [kW]	275	55

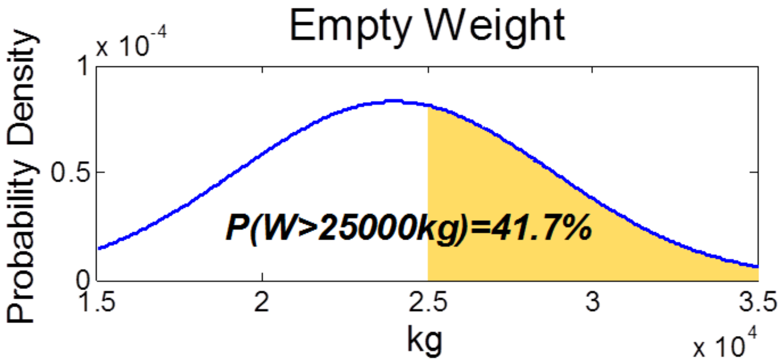
System Quantities of Interest

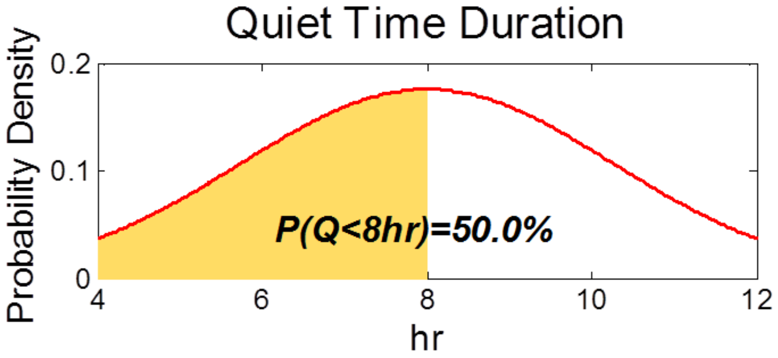
Quantity of Interest (QOI)		
1.	**R**	Range [km]
2.	**W**	Empty weight [kg]
3.	**Q**	Quiet time duration [hr]
4.	**A**	Maneuver Acceleration [# pulses]

Decomposition of Design Problem



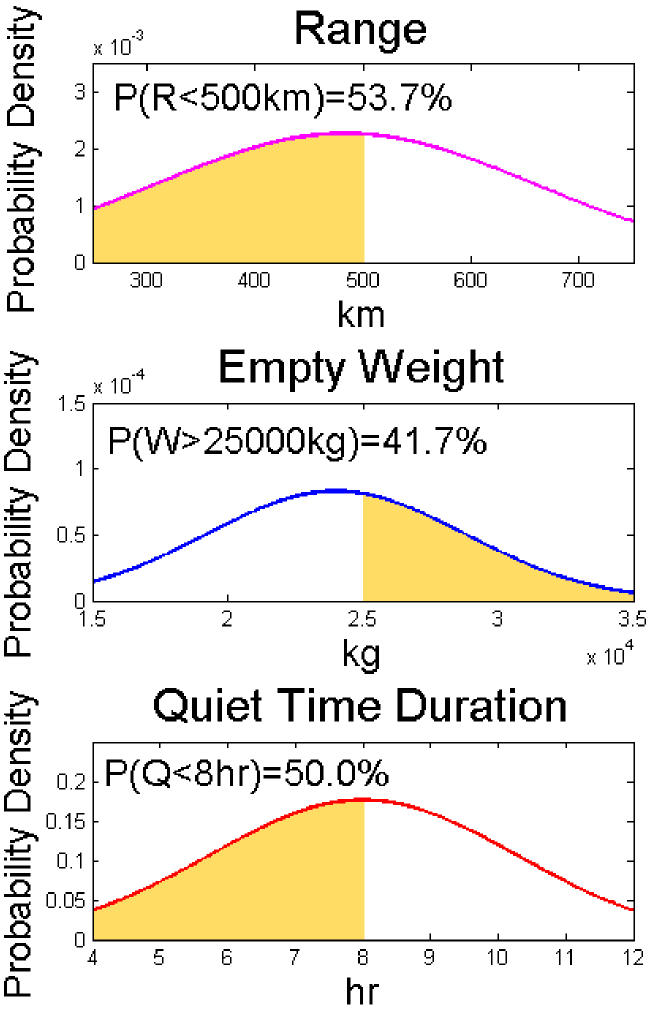


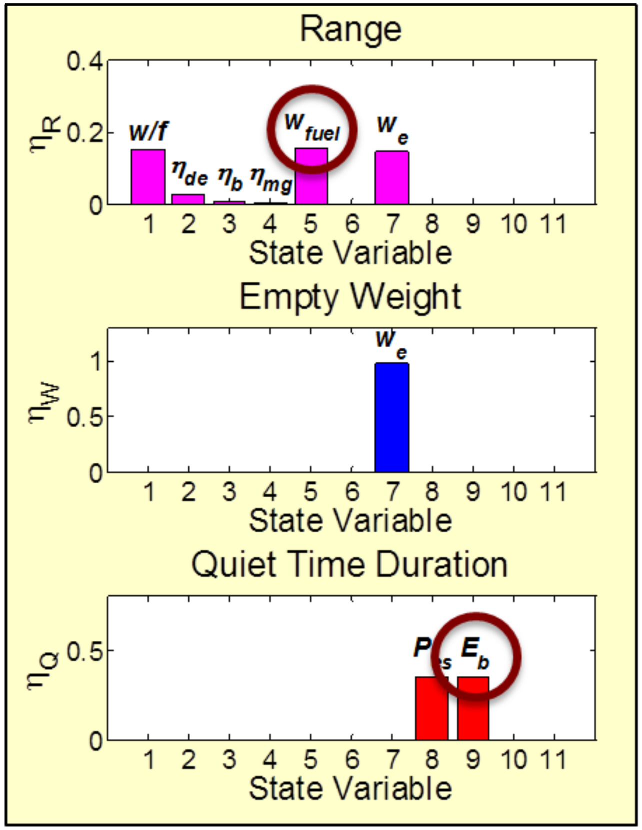


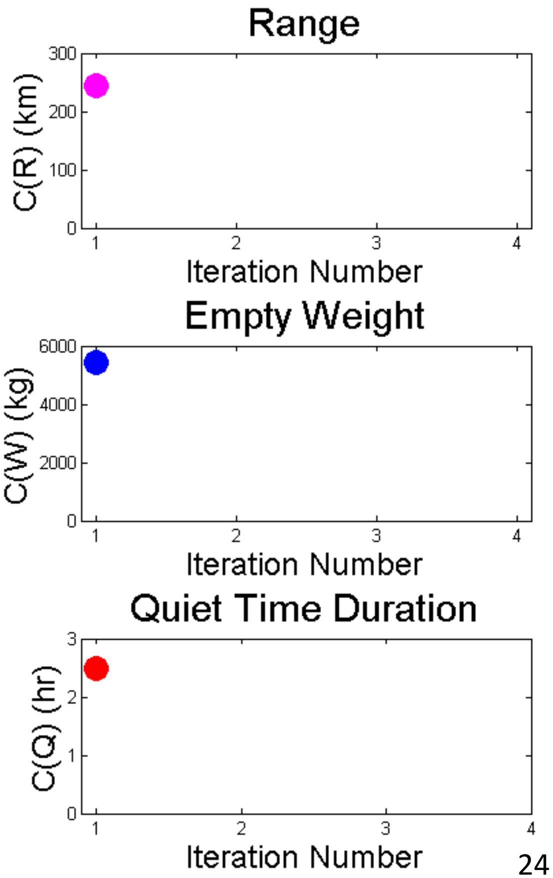


Iteration 1

- Initialize values and standard deviations
- Identify W_{fuel} and E_b as targets for resource allocation

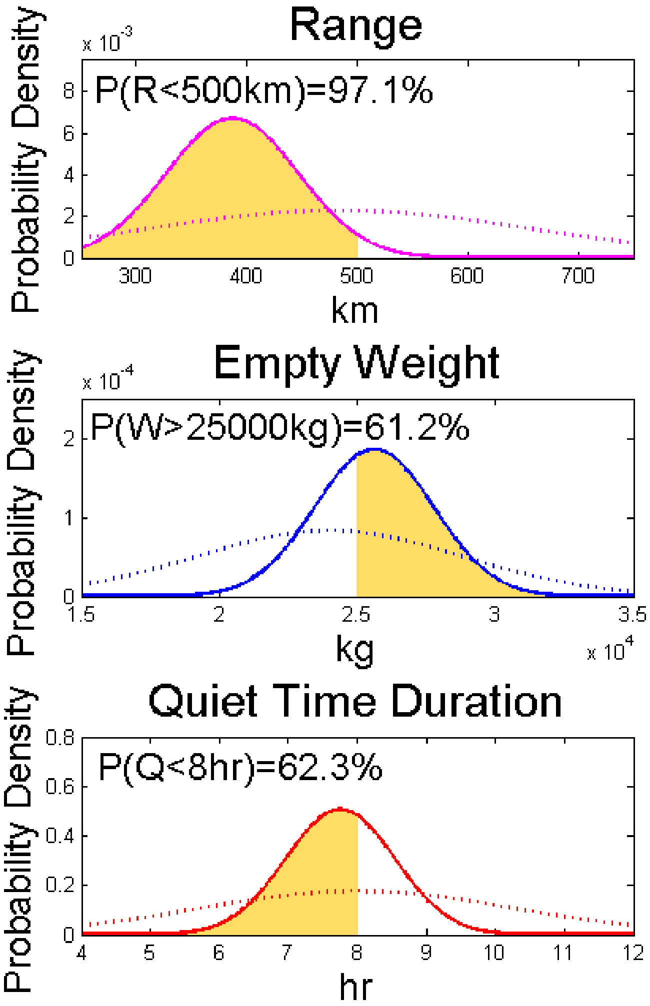


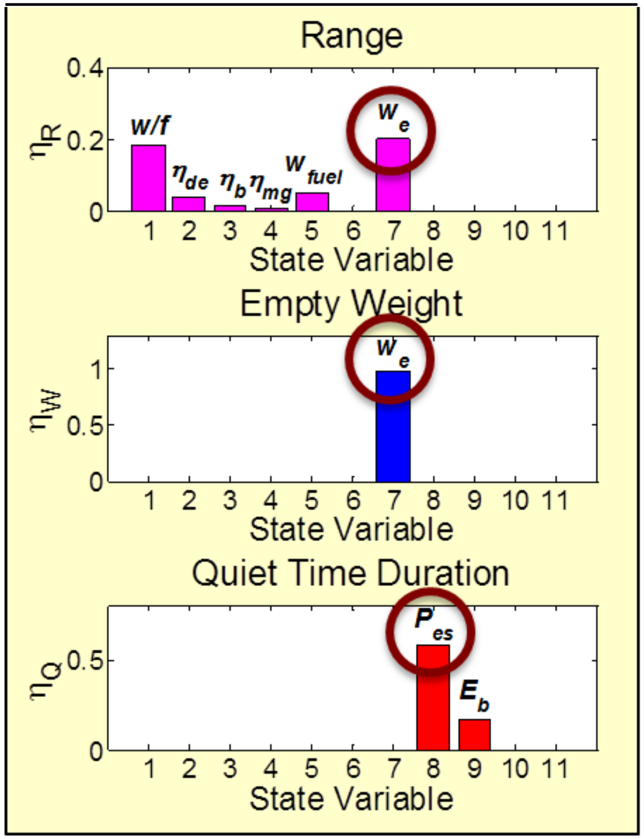


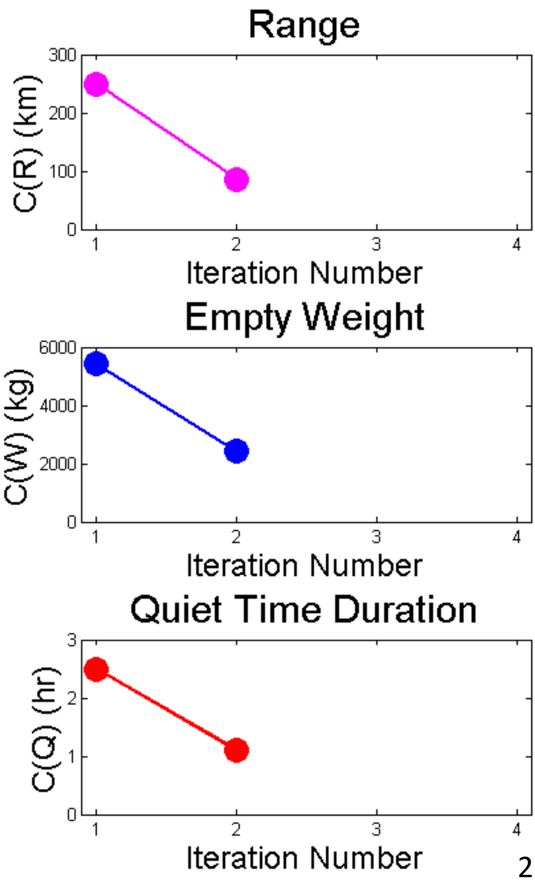


Iteration 2

- Reduce std. dev. of W_{fuel} and E_b by 75%; all others by 50%
- Identify W_e and P_{es} as targets for resource allocation

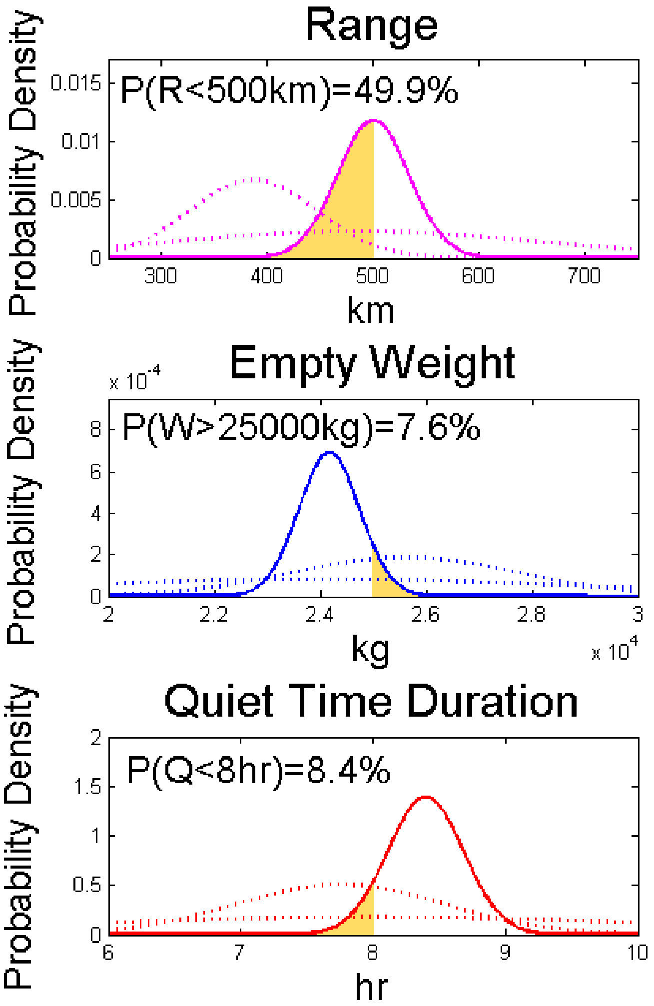


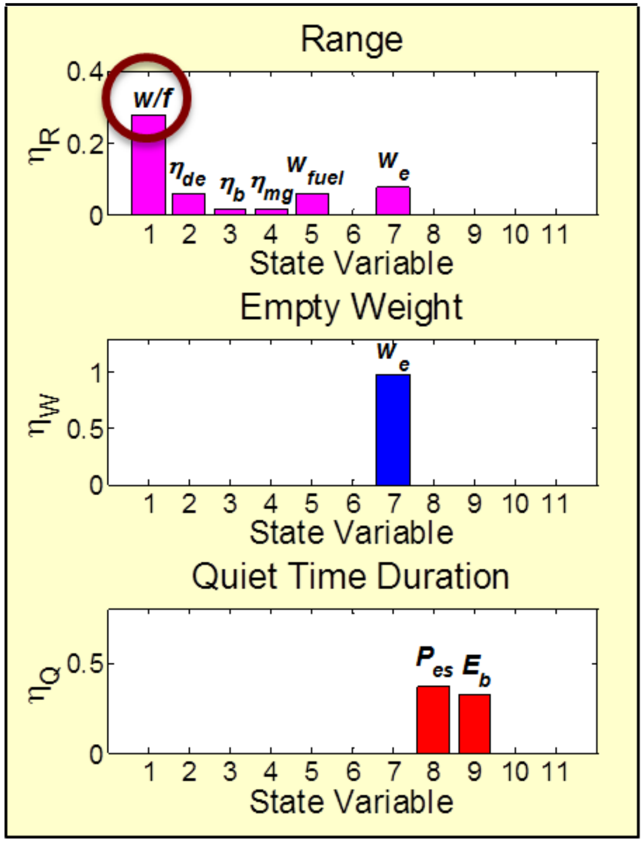


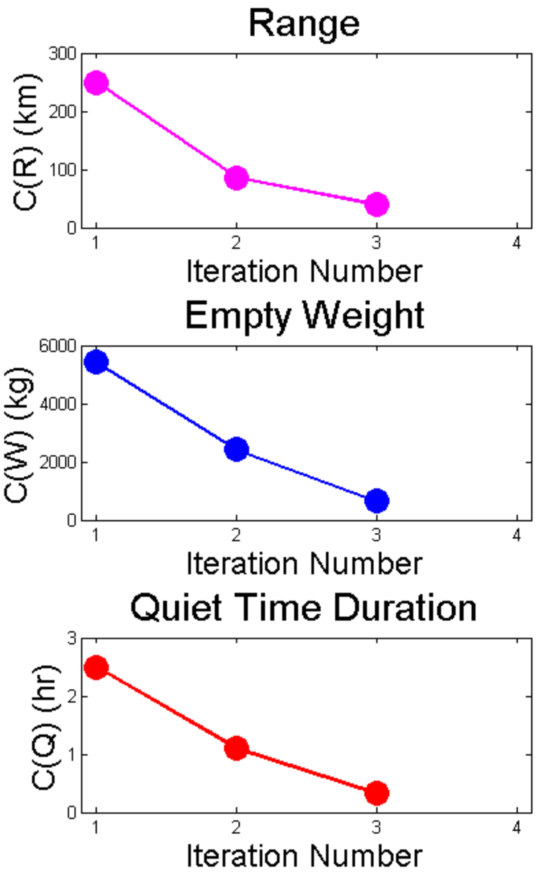


Iteration 3

- Reduce std. dev. of W_e and P_{es} by 75%; all others by 50%
- Identify w/f for resource allocation

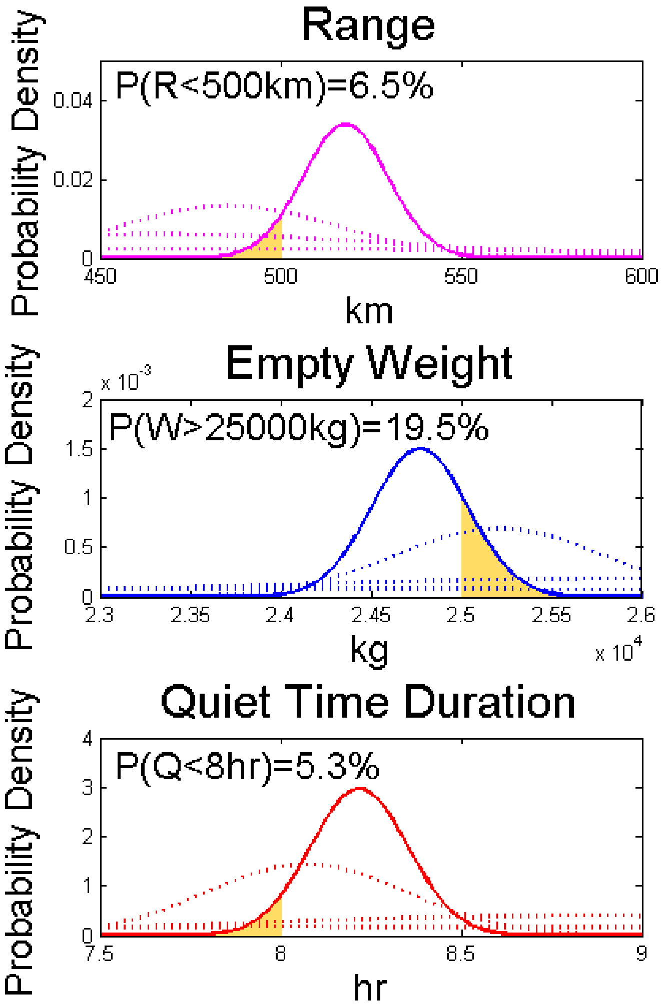


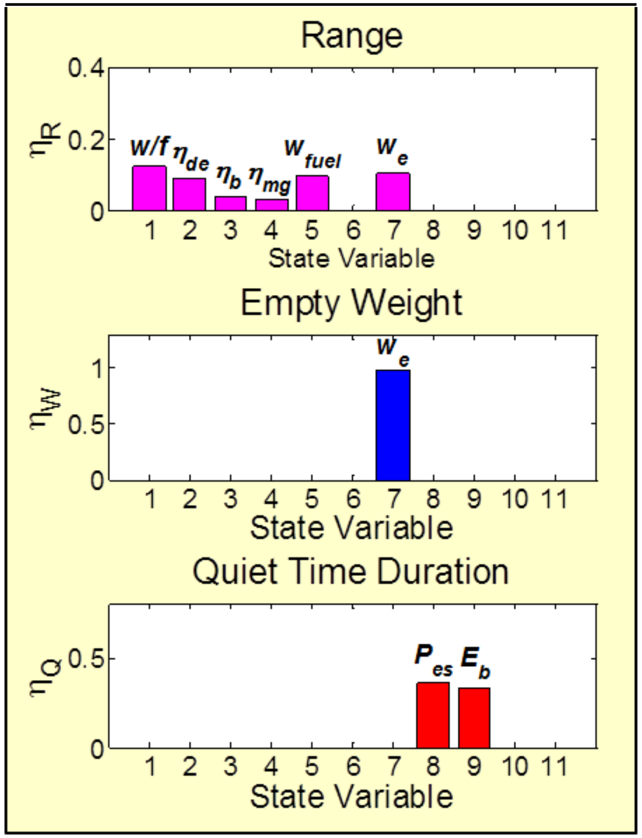


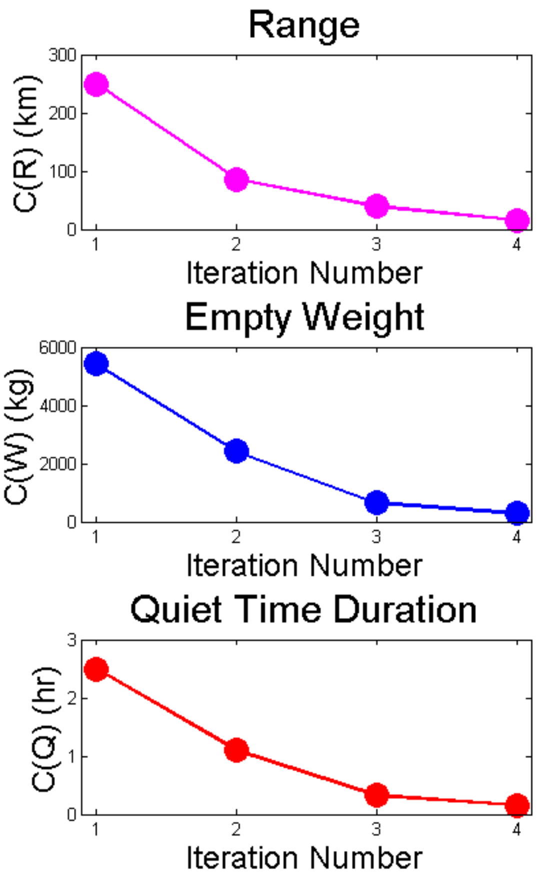


Iteration 4

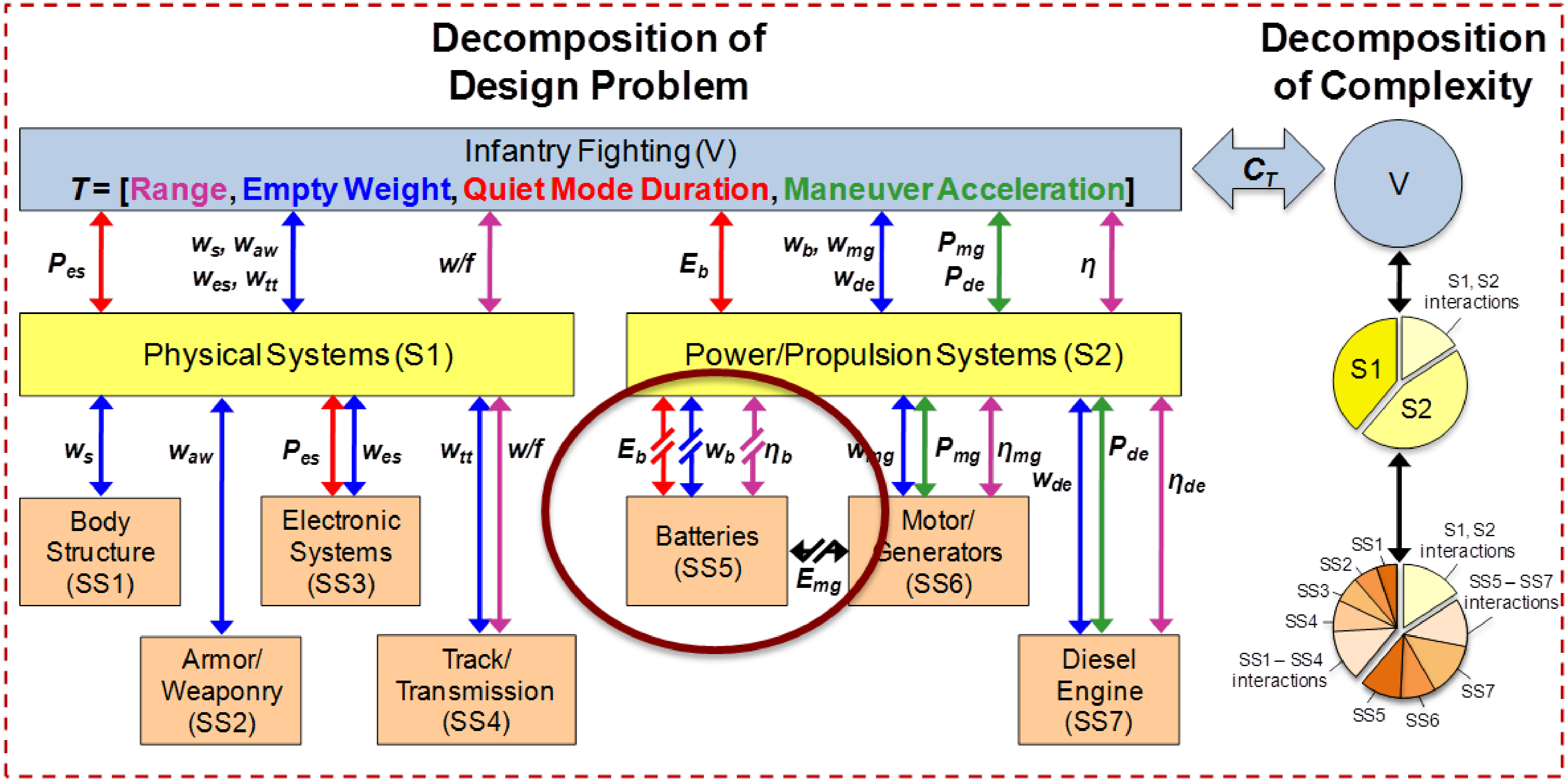
- Reduce std. dev. of w/f by 75%; all others by 50%







The Importance of Feedback



Decomposition of Design Problem

Decomposition of Complexity

Infantry Fighting (V)

T **= [Range, Empty Weight, Quiet Mode Duration, Maneuver Acceleration]**

Physical Systems (S1)

Power/Propulsion Systems (S2)

Body Structure (SS1)

Armor/ Weaponry (SS2)

Electronic Systems (SS3)

Track/ Transmission (SS4)

Batteries (SS5)

Motor/ Generators (SS6)

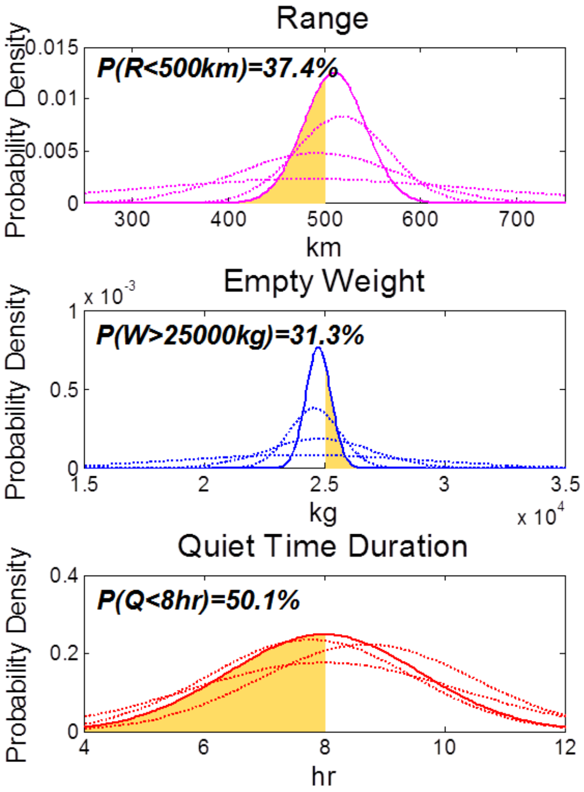
Diesel Engine (SS7)

Iterate until feasible design achieved

Broken Battery Feedback Results

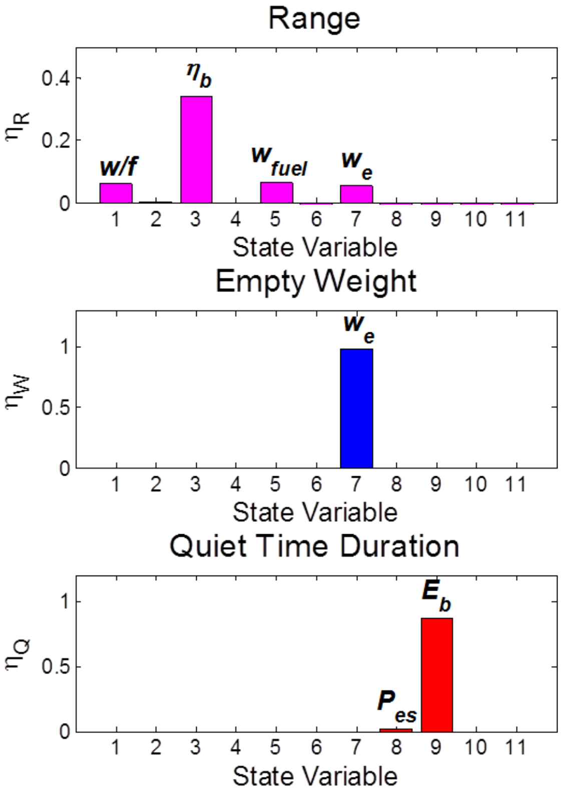
QOI Estimates

Uncertainty in QOI estimates is slower to decrease between iterations



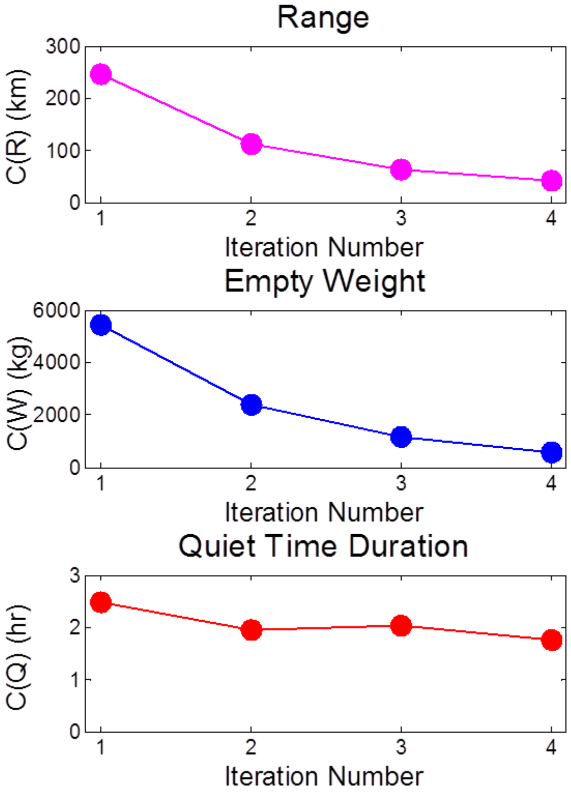
Sensitivity Indices

Battery-related state variables (**η_b**, **E_b**) contribute greatly to complexity in QOI



Complexity Estimates

Complexity in QOI estimates is slower to converge



Relevance of the Example

Source: www.boeing.com.

Source: blog.seattlepi.com.

Conclusions

- Summary
  - Proposed an information-theoretic metric of complexity
  - Developed a set of sensitivity indices as indicators of key sources of complexity
  - Calculated the metric of complexity and apportioned the complexity to key sources for an IFV application
  - Demonstrated the importance of feedback in design
- Conclusions
  - For simulation-based design and analysis, all sources of uncertainty must be included
  - Data regarding model discrepancy is critical
  - Sensitivity analysis can be used to allocate resources aimed at reducing large uncertainties in quantities of interest
  - The quantification and evolution of information in system design is essential.
  - System design and analysis is a problem of information management / uncertainty control
- Future work
  - Information fusion (models, sensors, experts…)
  - Compositional UQ

Bringing high fidelity forward

"The most important development in aviation in 2011" -Time

- Aerospace vehicle design typically involves custom parts for nearly every aspect of the system
- Design options include:
  - Use high fidelity tools to analyze
  - Start with low fidelity tools and identify where fidelity increases are required
  - Deal with emergent behavior as it emerges

Source: www.time-az.com.

- Reuse of parts/components enables high fidelity results from "low" fidelity tools
  - Sacrifice optimality for reduced complexity designs
  - Possibly at lower cost and faster development times
- Recall visualization discussion
  - Visualizing high dimensional design parameter spaces is difficult
  - Lots of room for possibly undetected emergent behavior
  - Foundry-like approach can reduce the design space substantially

References

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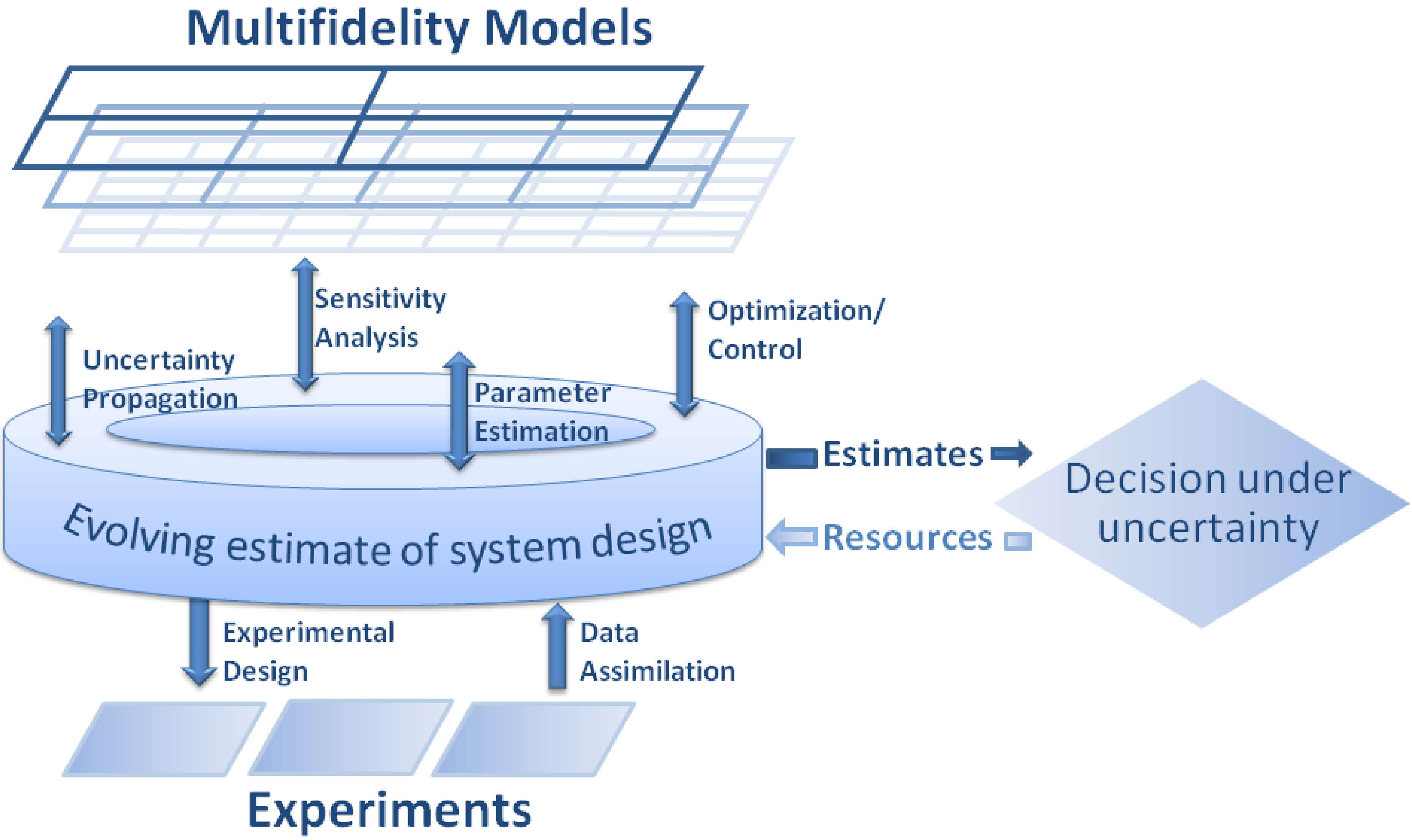
Big Picture


Multifidelity Models
Sensitivity Analysis
Uncertainty Propagation
Parameter Estimation
Optimization/ Control
Evolving estimate of system design
Estimates
Resources
Decision under uncertainty
Experimental Design
Data Assimilation
Experiments