COMER

# ECE 600 Homework 8

1. A message requires $N$ time units to be transmitted, where $N$ is a geometric random variable with pmf $p_j = (1-a)a^{j-1}$ for $j = 1, 2, \ldots$. A single new message arrives during a time unit with probability $p$, and no new messages arrive with probability $1-p$. Let $K$ be the number of new messages that arrive during the transmission of a single message.

   (a) Find the pmf of $K$. *Hint:* $(1-\beta)^{-(k+1)} = \sum_{n=k}^{\infty} \binom{n}{k} \beta^{n-k}$.

   (b) Fine $\mathrm{E}[K]$ and $\mathrm{Var}[K]$ using iterated expectation.

2. For any two random variables $X$ and $Y$ with $\mathrm{E}[X^2] < \infty$, show that $\mathrm{Var}(X) \geq \mathrm{E}[\mathrm{Var}(X|Y)]$, where $\mathrm{Var}(X|Y) = \mathrm{E}[X^2|Y] - (\mathrm{E}[X|Y])^2$.

3. The input $X$ to a communication channel is +1 or -1 with probability $p$ and $1-p$, respectively. The received signal $Y$ is the sum of $X$ and noise $N$ which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.25$.

   (a) Find the joint probability $P(X = j, Y \leq y)$.

   (b) Find the marginal pmf of $X$ and the marginal pdf of $Y$.

   (c) Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

4. The input $X$ to a communication channel is a zero-mean, unit-variance Gaussian random variable. The channel output $Y$ is the sum of $X$ and a noise signal $N$, where $N$ is a zero-mean Gaussian random variable with variance $\sigma_N^2$. The random variables $X$ and $N$ are independent.

   (a) Find the conditional pdf of $Y$ given $X = x$. *Hint:* $Y = N + x$ is a linear function of $N$.

   (b) Find the joint pdf of $X$ and $Y$.

   (c) Find the conditional pdf of $X$ given $Y = y$.

   (d) Suppose that when $Y = y$ we estimate the input $X$ by the value $x_0 = g(y)$ that maximizes $P(x_0 < X < x_0 + dx | Y = y)$. Find $x_0$.

5. Let $X_1, \ldots, X_n$ be random variables that are independent with finite variances. Form

$$Y = \sum_{i=1}^{n} \alpha_i X_i$$

   where $\alpha_i$ is real for every $i$. Find the mean and variance of $Y$.

6. Let $X_1, X_2, ..., X_n$ be $n$ i.i.d. random variables having uniform distribution on [0,1]. Also, let $X_{(1)}$ and $X_{(n)}$ be random variables denoting the minimum and maximum of the $X_i$'s, respectively. We define the random variable $Y = X_{(1)} + X_{(n)}$. Compute the pdf of $Y$.

7. Let $\omega$ be selected from the interval $\mathcal{S} = [0, 1]$, where the probability that $\omega$ is in a subinterval of $\mathcal{S}$ is equal to the length of the subinterval. For $n \geq 1$, define the following random sequences:

   (a) $U_n(\omega) = \frac{\omega}{n}$

   (b) $V_n(\omega) = \omega\left(1 - \frac{1}{n}\right)$

   (c) $W_n(\omega) = \omega e^n$

   (d) $Y_n(\omega) = \cos(2\pi n\omega)$

   (e) $Z_n(\omega) = e^{-n(nw-1)}$

   Which of these sequences converges everywhere? Almost everywhere? Identify the limiting random variable.

8. Let $X_n$ and $Y_n$ be two (possibly dependent) sequences of random variables that converge in the mean square sense to $X$ and $Y$, respectively. Does the sequence $X_n + Y_n$ converge in the mean square sense, and if so, to what limit?

9. Show that if $a_n \to a$ and $\mathrm{E}[|X_n - a_n|^2] \to 0$, then $X_n \to a$ in the mean square sense as $n \to \infty$.