COMER

ECE 600 Homework 7

1. Show that if ρ is the correlation coefficient for X and Y, then $|\rho| \leq 1$.

2. Show that if X and Y are uncorrelated random variables, then the variance of their sum is the sum of their variances, i.e., $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$.

3. Two random variables are said to be equal in the mean square sense if $\mathrm{E}[(X-Y)^2] = 0$. Show that if $\rho = 1$, where ρ is the correlation coefficient, then there exist real numbers a and b such that $Y = aX + b$ in the mean square sense.

4. The hat-check staff has had a long day, and at the end of the party they decide to return people's hats at random. Suppose that n people have their hats returned at random. You have previously shown that the expected number of people who get their own hat back is 1, irrespective of the total number of people. In this problem you are asked to find the variance of the number of people who get their own hat back. Let $X_i = 1$ if person i gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$ be the total number of people who get their own hat back.

   (a) Show that $\mathrm{Var}(S_n) = 1$.

   (b) Explain why you cannot use the variance of sums formula to calculate $\mathrm{Var}(S_n)$.

5. Let us consider four zero-mean, unit-variance random variables T, U, V, and W. Assume that they are pairwise-uncorrelated. We next define new random variables X, Y, and Z as $X = T + U$, $Y = U + V$, and $Z = V + W$. Compute the correlation coefficient between X and Y, and between X and Z.

6. Let X and Y be independent random variables with variances σ_X^2 and σ_Y^2. Consider the sum

$$Z = aX + (1-a)Y$$

   where $0 \leq a \leq 1$. Find a that minimizes the variance of Z.

7. Show that if X is a Cauchy random variable with parameter α, then

$$\Phi_X(\omega) = e^{-\alpha|\omega|}$$

   where $\Phi_X(\omega) = \mathrm{E}[e^{j\omega x}]$ is the characteristic function of X.

8. Let X be a random variable with characteristic function $\Phi_X(\omega)$. Show that $|\Phi_X(\omega)|$ takes on its maximum value at $\omega = 0$.