ECE 600 Homework 2

Note: Problems 7 and 9 are optional problems. They are included as they may be good for you to attempt as practice problems, but you are not required to complete them.

1. Let A and B be two events in a probability space.
   (a) Show that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.
   (b) Show that if $\mathrm{P}(A) = \mathrm{P}(B) = 1$, then $\mathrm{P}(A \cap B) = 1$.
   (c) Show that the probability that exactly one of the events A or B occurs is given by $\mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B)$.
   (d) Show that if A is a subset of B, then $\mathrm{P}(A) \leq \mathrm{P}(B)$.

2. Let A and B be two events in a probability space.
   (a) Show that if A and B are independent, then A^C and B are also independent.
   (b) If A and B are independent, are they mutually exclusive? Explain.
   (c) Show that if $\mathrm{P}(A|B) > \mathrm{P}(A)$, then $\mathrm{P}(B|A) > \mathrm{P}(B)$.

3. IF A, B, and C are events in a probability space, show that $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A|B{\cap}C)\mathrm{P}(B|C)\mathrm{P}(C)$.

4. An experiment consists of picking one of two urns at random, with the two urns being equally likely, and then selecting a ball from the urn and noting its color (black or white). Let A be the event "urn 1 is selected" and B the event "a black ball is selected". Under what conditions are the events A and B independent?

5. There are two servers at the grocery store. Server 1 can take anywhere from 1 to 5 minutes to complete an order. Server 2 can take anywhere from 1 to 10 minutes to complete an order. Let T_S denote the service time. You are given that

$$\mathrm{P}(T_S \leq t|\text{Server 1}) = \begin{cases} 0 & t \leq 1 \\ \frac{t-1}{4} & 1 < t \leq 5 \\ 1 & t > 5 \end{cases}$$

$$\mathrm{P}(T_S \leq t|\text{Server 2}) = \begin{cases} 0 & t \leq 1 \\ \frac{t-1}{9} & 1 < t \leq 10 \\ 1 & t > 10 \end{cases}$$

You are also given that $\mathrm{P}(\{\text{Server 1}\}) = 0.4$ and $\mathrm{P}(\{\text{Server 2}\}) = 0.6$.

(a) Find $\mathrm{P}(\{T_S \leq 5\})$.

(b) Find $\mathrm{P}(\{\text{Server } 1\} | \{T_S \leq 5\})$.

(c) Find $a > 1$ such that $\mathrm{P}(\{\text{Server } 1\} | \{T_S \leq a\}) = \mathrm{P}(\{\text{Server } 2\} | \{T_S \leq a\})$. This means that given $T_S \leq a$, it is equally likely that either server performed this task.

6. For events $A_1, ..., A_n$, consider the 2^n equations

$$\mathrm{P}(B_1 \cap B_2 \cap ... \cap B_n) = \mathrm{P}(B_1)\mathrm{P}(B_2)...\mathrm{P}(B_n)$$

with $B_i = A_i$ or $B_i = A_i^C$ for each i. Show that $A_1, ..., A_n$ are independent if all these equations hold.

7. Note: This problem is optional. If you choose to attempt it for practice, you do not need to solve the problem completely, but should simply try to set up a solution.

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space. The collection $\mathscr{C}_i, i = 1, ..., n$ of sets in $\mathscr{F}$ are said to be independent iff given any choice of C_i in $\mathscr{C}_i$ for every $i = 1, ..., n$ the events C_i are independent. Show that if $\mathscr{C}_i, i = 1, ..., n$ are independent, then if the following sets are added to each $\mathscr{C}_i$, the new collections still remain independent.

(a) Set differences $A - B$, for every A, B in $\mathscr{C}_i$ such that B is a subset of A.

(b) Countable disjoint unions of sets in $\mathscr{C}_i$.

Also, give an example that shows that finite intersections cannot be added to create new independent collections.

8. You are a contestant on a TV game show. There are 3 identical closed doors leading to 3 rooms. Two of the rooms contain nothing, but the third contains a brand new Mercedes which is yours if you make the correct choice. You are asked to pick a door by the host of the show who knows which room contains the Mercedes. Once you have made your decision, he shows you a room (not the one you chose) that does not contain the car. Show that even without any further knowledge, you will greatly increase your chances of winning the Mercedes if you switch your choice to the other door still unopened.

9. Note: This problem is optional.

Show that if $\mathrm{P}(A_i) = 1$ for all $i \geq 1$, then $\mathrm{P}(\bigcap_{i=1}^{\infty} A_i) = 1$.