COMER
~~November~~ 12, 2017
December

ECE 600 Final Exam

1. Enter your name and signature in the space provided below. Your signature indicates that you have not received any assistance from notes or other references, or from other students, during the exam.
2. You may not use a calculator or any other reference materials.
3. Partial credit will be given, at the discretion of the instructor. You must clearly justify your solution to receive full credit.

Name:

SOLUTION

Signature:

1. (25 points) Suppose that the joint density function of the random variables X and Y is given by

$$f_{XY}(x,y) = \frac{e^{-x/y}e^{-y}}{y}$$

for $0 < x < \infty, 0 < y < \infty$.

(a) Find the marginal density function of Y.

(b) Use iterated expectation to find $E[X]$.

$$(a)\ f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}\,dx = \frac{e^{-y}}{y}(-y)e^{-x/y}\Big|_0^\infty$$

$$f_Y(y) = e^{-y}, \quad 0<y<\infty$$

$$(b)\ E[X] = E[E[X|Y]]$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-x/y}}{y}$$

$$E[X|Y=y] = \int_0^{+\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_0^\infty \frac{x}{y}e^{-x/y}\,dx = y$$

$$E[X|Y] = Y$$

So $E[X] = \cancel{E[X]}\ \cancel{E[X} \ E[E[X|Y]] = E[Y]$

$$E[X] = \int_0^{+\infty} y e^{-y}\,dy = 1$$

2. (25 points) Consider a random variable X, and let c_n be a sequence of positive real numbers converging to a limit c. Show that if $Y_n = c_n X$, then Y_n converges in distribution to $Y = cX$.

$$F_{Y_n}(y) = P(Y_n \le y) = P(c_n X \le y) =$$

$$P\left(X \le \frac{y}{c_n}\right) = F_X\left(\frac{y}{c_n}\right)$$

For $c > 0$: $F_Y(y) = P(cX \le y) = F_X\left(\frac{y}{c}\right)$

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} F_X\left(\frac{y}{c_n}\right) = F_X\left(\lim_{n\to\infty} \frac{y}{c_n}\right)$$

$$= F_X(y/c) = F_Y(y)$$

So $Y_n \to Y = cX$ in distr. if $c > 0$

For $c = 0$: $F_Y(y) = P(0 \cdot X \le y) = \begin{cases} 1 & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}$

$$\lim_{n\to\infty} F_X\left(\frac{y}{c_n}\right) = F_X\left(\lim_{n\to\infty} \frac{y}{c_n}\right) =$$

$$\begin{cases} F_X(\infty) = 1 & \text{if } y > 0 \\ F_X(-\infty) = 0 & \text{if } y < 0 \end{cases} = F_Y(y)$$

Note: if $c = 0, y = 0$, $F_Y(0) = F_X(0)$

3. (25 points) Consider a random process $D(t) = X(t) - X(t-d)$, where $X(t)$ is a WSS process and $d > 0$ is a real number. If $D(t)$ is passed through a LTI system with impulse response $h(t)$, find the mean and autocorrelation function of the output of the system in terms of $R_X(\tau)$ and $h(t)$.

$$D(t) = X(t) - X(t-d) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$E[Y(t)] = E\left(\int_{-\infty}^{\infty} h(t)X(t)\,dt - \int_{-\infty}^{\infty} h(t)X(t-d)\,dt\right]$$

$$= \int h(t)E[X(t)]\,dt - \int h(t)E[X(t-d)]\,dt = 0 - 0$$

$$\underline{E[Y(t)] = 0}$$

(Could also use $E[D(t)] = \underbrace{E[X(t) - X(t-d)]}_{=0}$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(t-\alpha)\mu_D(\alpha)\,d\alpha = 0 \;)$$

First find

$R_{DD}(t_1,t_2)$: $E[D(t_1)D(t_2)] = E[[X(t_1) - X(t_1-d)][X(t_2) - X(t_2-d)]]$

$$= E(X(t_1)X(t_2)) - X(t_1-d)X(t_2) - X(t_1)X(t_2-d) + X(t_1-d)X(t_2-d)]$$

$$= R_X(\tau) - R_X(\tau+d) - R_X(\tau-d) + R_X(\tau)]$$

$$= 2R_X(\tau) - R_X(\tau+d) - R_X(\tau-d)$$

So $D(t)$ is WSS. This means that $Y(t)$ is WSS, and

$$R_Y(\tau) = (\tilde{h} * h * R_D)(\tau), \quad \tilde{h}(t) = h(-t)$$

So $\cancel{\tilde{h} * h * (2R_X(\tau) - R_X(\tau+d) - R_X(\tau-d))}$

$$R_Y(\tau) = \tilde{h}(\tau) * h(\tau) * [2R_X(\tau) - R_X(\tau+d) - R_X(\tau-d)]$$

4. (25 points) Let $X(t), t \geq 0$ be a random process with mean function $\mu_X(t) = \mu_0$ and autocovariance function $C_{XX}(t_1, t_2) = \sigma_X^2(min(t_1, t_2))$. For a fixed $T > 0$, find the mean and autocovariance function of $Y(t) = X(t) - X(T)$ for all $t \geq T$. Note that for the autocovariance function of $Y(t)$, you need only consider times t_1 and t_2 that are both greater than T.

$$Y(t) = X(t) - X(T)$$

$$E[Y(t)] = E[X(t) - X(T)] = \mu_0 - \mu_0 = 0$$

$C_{YY}(t_1, t_2) = R_{YY}(t_1, t_2)$, since $\mu_Y(t) = 0$

So $C_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] =$

$$E[(X(t_1) - X(T))(X(t_2) - X(T)] =$$

$$E[X(t_1)X(t_2)] - E[X(t_1)X(T)] -$$

$$E[X(T)X(t_2)] + E[X(T)X(T)] =$$

$$R_{XX}(t_1, t_2) - R_{XX}(t_1, T) - R_{XX}(T, t_2) + R_{XX}(T, T)$$

Now $R_{XX}(t_1, t_2) = C_{XX}(t_1, t_2) + \mu_0^2$, so

$$C_{YY}(t_1, t_2) = C_{XX}(t_1, t_2) + \mu_0^2 - C_{XX}(t_1, T) - \mu_0^2$$

$$- C_{XX}(T, t_2) - \mu_0^2 + C_{XX}(T, T) + \mu_0^2$$

$$= C_{XX}(t_1, t_2) - C_{XX}(t_1, T) - C_{XX}(T, t_2) + C_{XX}(T, T)$$

$$= \sigma_X^2(min(t_1, t_2)) - \sigma_X^2(T) - \sigma_X^2(T) + \sigma_X^2(T)$$

So $C_{YY}(t_1, t_2) = \sigma_X^2(min(t_1, t_2)) - \sigma_X^2(T)$