COMER
November 10, 2015

ECE 600 Exam 2

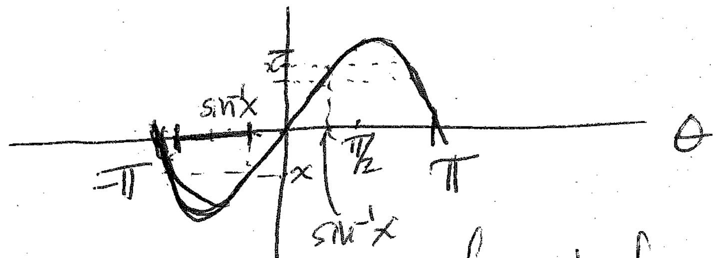
1. Enter your name and signature in the space provided below. Your signature indicates that you have not received any assistance from notes or other references, or from other students, during the exam.
2. You may not use a calculator or any other reference materials.
3. Partial credit will be given, at the discretion of the instructor. You must clearly justify your solution to receive full credit.

Name: SOLUTION

Signature:

1. (25 points) Let Θ be a continuous random variable that is uniformly distributed on $[-\pi, \pi]$ and let $X = \sin \Theta$. Find the cdf of X.

$g(\theta) = \sin\theta, \quad -\pi \le \theta \le \pi$



Let $\sin^{-1}x$ be the value of $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $x = \sin\theta$.

$R_X = [-1, 1]$

For $-1 \le x < 0$:

$F_X(x) = P(X \le x)$

$= P(-\pi - \sin^{-1}x \le \Theta \le \sin^{-1}x) =$

$$\int_{-\pi-\sin^{-1}x}^{\sin^{-1}x} \frac{1}{2\pi}\, d\Theta = \frac{1}{2\pi}\left[\sin^{-1}x + \pi + \sin^{-1}x\right]$$

$$= \frac{2\sin^{-1}x + \pi}{2\pi}$$

For $0 \le x \le 1$:

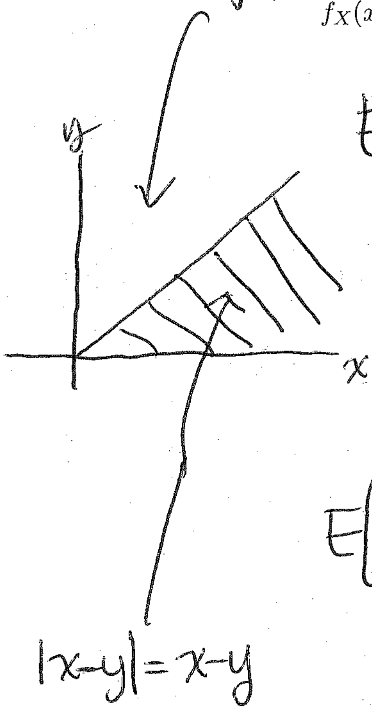
$F_X(x) = P(-\pi \le \Theta < 0) + P(0 \le \Theta < \sin^{-1}x) + P(\pi - \sin^{-1}x \le \Theta \le \pi)$

$= \frac{1}{2} + \frac{\sin^{-1}x}{2\pi} + \frac{\sin^{-1}x}{2\pi} =$

$\frac{1}{2} + \frac{2\sin^{-1}x}{2\pi}$

$|x-y| = y-x$

2. (25 points) If X and Y are independent exponential random variables with means μ_X and μ_Y, find $E[|X-Y|]$. Note that an exponential random variable X with mean μ has probability density function $f_X(x) = \frac{1}{\mu}exp(-\frac{x}{\mu})$.



$$E[|X-Y|] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{XY}(x,y)\,dx\,dy$$

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{\mu_X} e^{-x/\mu_X} \cdot \frac{1}{\mu_Y} e^{-y/\mu_Y} u(x)u(y)$$

$$E[|X-Y|] = \int_0^{\infty}\int_0^{x} \frac{x-y}{\mu_X\mu_Y} e^{-x/\mu_X} e^{-y/\mu_Y}\,dy\,dx +$$

$$\int_0^{\infty}\int_x^{\infty} \frac{y-x}{\mu_X\mu_Y} e^{-x/\mu_X} e^{-y/\mu_Y}\,dy\,dx$$

$$= \frac{1}{\mu_X\mu_Y}\Big[\int_0^{\infty}\int_0^{x} x e^{-\mu_X} e^{-y/\mu_Y}\,dy\,dx -$$

$$\int_0^{\infty}\int_0^{x} y e^{-x/\mu_X} e^{-y/\mu_Y}\,dy\,dx + \int_0^{\infty}\int_x^{\infty} y e^{-x/\mu_X} e^{-y/\mu_Y}\,dy\,dx$$

$$- \int_0^{\infty}\int_x^{\infty} x e^{-x/\mu_X} e^{-y/\mu_Y}\,dy\,dx\Big]$$

Note: At this point, integration by parts can be used. You did not need to do this.

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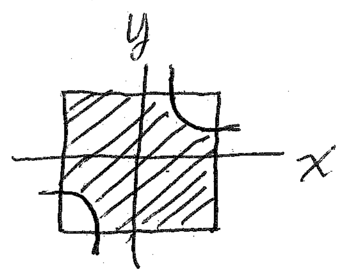
3. (25 points) Let X and Y be independent random variables that are uniformly distributed in $[-1, 1]$. Find the following probabilities:

(a) $P(XY < 1/2)$.

(b) $P(\max(X, Y) < 1/3)$.

$$f_{XY}(x,y) = \frac{1}{4}, \quad -1 \le x \le 1, -1 \le y \le 1$$



(a) $P\left(XY < \frac{1}{2}\right) = P\left(Y < \frac{1}{2X}\right)$

$= P\left((X,Y) \text{ lies in shaded area}\right)$

$$= 2\left[\int_{-1}^{1/2}\int_0^1 \frac{1}{4}\,dy\,dx + \int_{1/2}^{1}\int_0^{1/2x} \frac{1}{4}\,dy\,dx\right]$$

Using symmetry about the x axis.

$$= 2\left(\frac{1}{4}\right)\left(\frac{3}{2}\right)(1) + 2\left(\frac{1}{4}\right)\int_{1/2}^{1}\frac{1}{2x}\,dx$$

$$= \frac{3}{4} + \frac{1}{4}(\ln 2)$$

(b) $P\left(\max(X,Y) < \frac{1}{3}\right) = P\left(X < \frac{1}{3}, Y < \frac{1}{3}\right)$

$= P\left(X < \frac{1}{3}\right)P\left(Y < \frac{1}{3}\right)$ (independence of X, Y)

$$= \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)\left(\frac{4}{3}\right)\left(\frac{1}{2}\right) = \frac{4}{9}$$

4. (25 points) Let X and Y be two discrete random variables with joint probability mass function $p_{XY}(x_k, y_j)$ for $k = 0, 1, \ldots; j = 0, 1, \ldots$. We say that $X \geq Y$ if $P(X \geq Y) = 1$. Show that if $X \geq Y$, then $E[X] \geq E[Y]$.

Since we are given that $X \geq Y$, we have

$$P(X \geq Y) = P(\{(x_k, y_j) : x_k \geq y_j\}) = 1,$$

and also

$$P(\{(x_k, y_j) : x_k < y_j\}) = 0$$

We can write

$$E[X] = \sum_{k,j : x_k \geq y_j}\sum x_k\, p_{XY}(x_k, y_j) + \sum_{\substack{k,j: \\ x_k < y_j}}\sum x_k\, p_{XY}(x_k, y_j)$$

Now since

$$0 = P(\{(x_k, y_j) : x_k < y_j\}) = \sum_{\substack{k,j: \\ x_k < y_j}}\sum p_{XY}(x_k, y_j)$$

and $p_{XY}(x_k, y_j) \geq 0 \quad \forall j, k$,

we have

$$p_{XY}(x_k, y_j) = 0 \quad \forall j, k \text{ such that } x_k < y_j$$

so

$$E[X] = \sum_{\substack{k,j: \\ x_k \geq y_j}}\sum x_k\, p_{XY}(x_k, y_j)$$

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But this means that

$$E[X] \geq \sum_{\substack{k,j: \\ x_k \geq y_j}}\sum y_j \, p_{XY}(x_k, y_j)$$

Now we can also write

$$E[Y] = \sum_{\substack{j,k: \\ x_k \geq y_j}}\sum y_j \, p_{XY}(x_k, y_j) + \sum_{\substack{j,k: \\ x_k < y_j}}\sum y_j \, p_{XY}(x_k, y_j)$$

$$= \sum_{\substack{j,k: \\ x_k \geq y_j}}\sum y_j \, p_{XY}(x_k, y_j)$$

Thus $E[X] \geq E[Y]$