Conditional Probability

9/8/2022

We know that $\forall A \in \mathcal{F}$, there is a prob. $P(A)$. Often we want to know the prob. of event A given the occurrence of another event B. This is called a cond. prob.

Ex. Roll a fair die.

Let $S = \{1, 2, 3, 4, 5, 6\}$

Consider two events:

$A = \{2, 4, 6\}$, or A is the event "outcome is even"

$$P(A) = \frac{1}{2}$$

$B = \{4, 5, 6\}$, or "outcome > 3"

$$P(B) = \frac{1}{2}$$

What is the prob. of A given B occurs?

Intuitively, it is

$$\frac{|A \cap B|}{|B|} = \frac{2}{3}$$

To generalize this basic idea to make it work for any random exp., we use:

Defn. Given A and $B \in \mathcal{F}$, the ~~conditionaly~~ <u>conditional probability</u> of A given B is
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

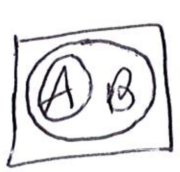
If $P(B) = 0$, $P(A|B)$ is undefined (in this class).

Note that $P(\cdot|B) : \mathcal{F} \to [0,1]$ is a valid prob. measure for any fixed $B \in \mathcal{F}$ with $P(B) > 0$.

Proof omitted.

Also, $P(\cdot|B)$ is part of the prob. space $(S, \mathcal{F}, P(\cdot|B))$

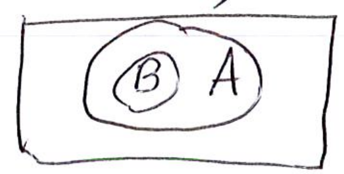
Some properties:



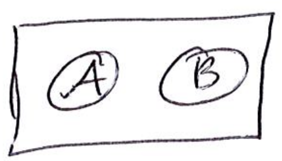
1. If $A \subset B$, then $P(A|B) = \dfrac{P(A)}{P(B)} \geq P(A)$

2. If $B \subset A$, then

$$P(A|B) = \frac{P(B)}{P(B)} = 1$$



3. If $A \cap B = \phi$, then



$$P(A|B) = \frac{P(\phi)}{P(B)} = 0$$

Bayes Theorem

Let $(S, \mathcal{F}, P)$ be a prob. space and $A, B \in \mathcal{F}$ with $P(A), P(B) > 0$.
We can write

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

from the defn. of cond. prob.
So $P(A|B)P(B) = P(B|A)P(A)$, or

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This is Bayes' Theorem

The Total Probability Law (TPL) (4)

Let $A_1, \ldots, A_n$, with $P(A_i) > 0 \ \forall A_i \in \mathcal{F}$, be a partition of S. Let $B \in \mathcal{F}$.

Then the TPL states that

$$P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$$

Proof. $P(B) = P(B \cap S)$

$$= P\left(B \cap \left(\bigcup_{i=1}^{n} A_i\right)\right)$$

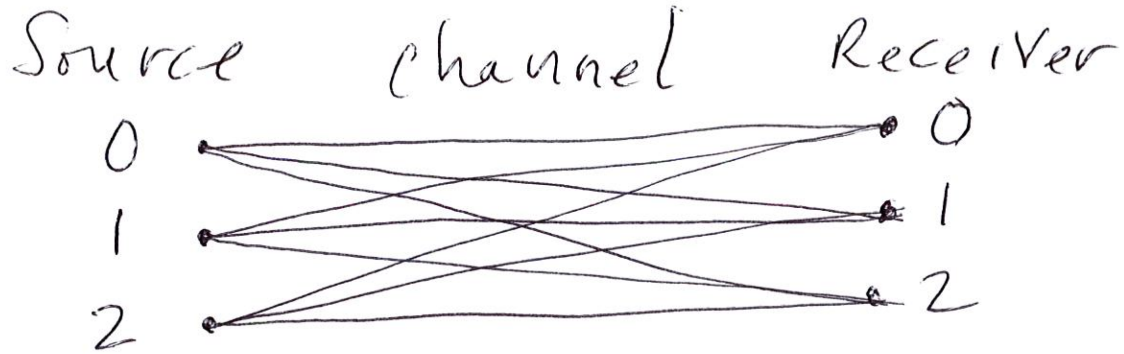
$$\cancel{= P(P(\bigcup_{i=1}} = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right)$$

(since $\cap$ distributes over union)

$$= \sum_{i=1}^{n} P(B \cap A_i)$$ from Axiom 3, since $A_1, \ldots, A_n$ are disjoint.

So $P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$ from the defn. of cond. prob.

Example. Consider a communications system with a 3-symbol alphabet $\{0,1,2\}$



Let $S = \{(i,j) : i$ is sent, j is received; $i,j \in \{0,1,2\}\}$

Consider two events:

$A = \{(0,0),(0,1),(0,2)\}$, or 0 is sent

$B = \{(0,0),(1,0),(2,0)\}$, or 0 is received.

Then $P(A|B) = P(0\ s \mid 0\ r)$

This prob. is what we want to know, but it is difficult to estimate directly.

Using Bayes' Theorem,
we have

$$P(Os|Or) = \frac{P(Or|Os)P(Os)}{P(Or)}$$

Now we need

- $P(Or|Os)$

  In practice, estimate this using "channel modeling"

- $P(Os)$

  Estimate this using "source modeling"

- $P(Or)$

  Use the TPL to write this in terms of the channel and source probs.

To find $P(Or)$:

let $A_i = \{(i,0),(i,1),(i,2)\}$ ⑦

Then A_0, A_1, A_2 form a partition

Then $P(0r) = \sum_{i=0}^{2} P(0r|is)P(is)$

Final result

$$P(0s|0r) = \frac{P(0r|0s)P(0s)}{\sum_{i=0}^{2} P(0r|is)P(is)}$$