9/29/2022

Looking at $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$. Find the distr. of Y from the distr. of X and the fn. g.

Consider 3 cases:

1. X, Y are both continuous

Approach 1. First find $F_Y(y)$ using

$$F_Y(y) = P(Y \le y) = P(g(X) \le y), \; y \in \mathbb{R}$$

Write as

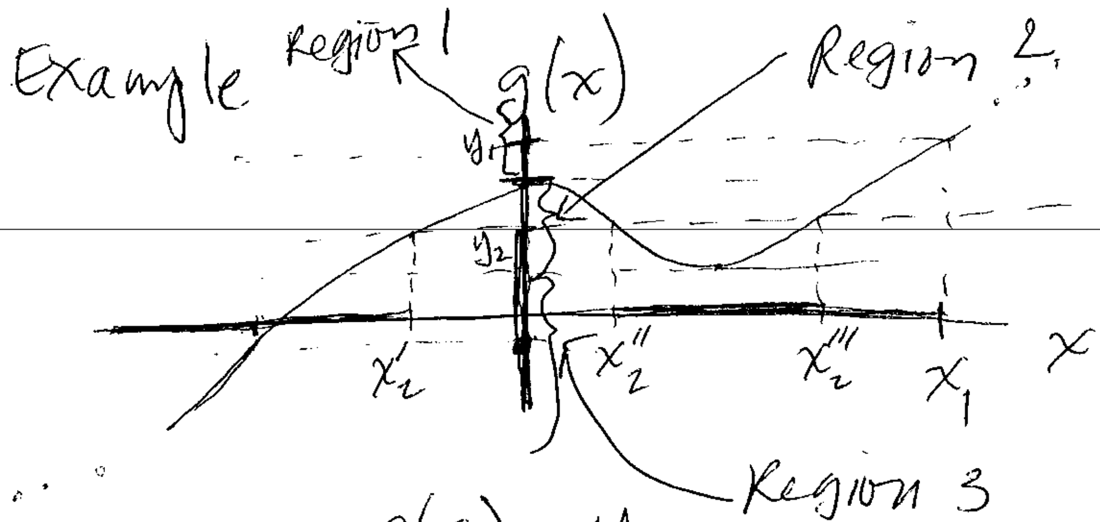
$$F_Y(y) = P(X \in D_y) \quad \text{for}$$

$$D_y = \{x \in \mathbb{R} : g(x) \le y\}$$

Note: $\{g(X) \le y\} = \{X \in D_y\}$ from the defn. of D_y, so $P(g(X) \le y) = P(X \in D_y)$

We need to find D_y for every $y \in \mathbb{R}$

Example



$$g(x_1) = y_1$$

$$g(x_2') = g(x_2'') = g(x_2''') = y_2$$

We see that $D_{y_1} = \{x : x \le x_1\}$,

since $\{Y \le y_1\} = \{X \le x_1\}$

So $P(Y \le y_1) = P(X \le x_1)$

For y_2, $P(Y \le y_2) = P(X \in D_{y_2})$

where $D_{y_2} = \{x : x \le x_2'\} \cup$

$\{x : x_2'' \le X \le x_2'''\}$

We can write $D_{y_1} = (-\infty, x_1]$ and

$D_{y_2} = (-\infty, x_2'] \cup [x_2'', x_2''']$

~~Can break up the vertical axis into three regions to find~~

So $F_Y(y_1) = \int_{-\infty}^{x_1} f_X(x)\,dx$ and (3)

$$F_Y(y_2) = \int_{-\infty}^{x_2'} f_X(x)\,dx + \int_{x_2''}^{x_2'''} f_X(x)\,dx$$

Can break up the vertical axis into three regions to find $F_Y(y)$ $\forall y \in \mathbb{R}$

Then let $f_Y(y) = \frac{d}{dy} F_Y(y)$

Note:

$$F_X(x) = P(X \leq x)$$

What about $P(X < x)$?

Write $P(X \leq x) = P(\{X < x\} \cup \{X = x\})$

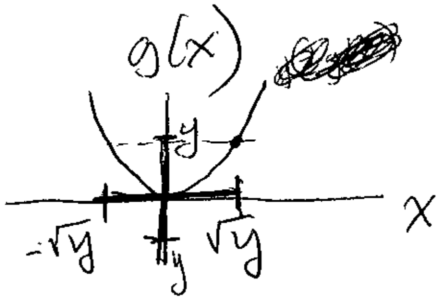
$$= P(X < x) + \underbrace{P(X = x)}_{0 \text{ if}}$$

X is continuous

Example. Let $Y = X^2$.

Have $g(x) = x^2$

Note that this does not mean $y = g(x)$ or $y = x^2$.



For $y > 0$,

$$P(Y \le y) = P(X \in D_y)$$

where $D_y = [-\sqrt{y}, \sqrt{y}]$

So $F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$ for $y > 0$

For $y < 0$, $F_Y(y) = P(\phi) = 0$

since $\{x : x^2 < 0\} = \phi$

For $y = 0$, $F_Y(y) = P(X = 0) = 0$

Then $$f_Y(y) = \begin{cases} \dfrac{f_X(-\sqrt{y})}{2\sqrt{y}} + \dfrac{f_X(\sqrt{y})}{2\sqrt{y}}, & y > 0 \\ 0 & y \le 0 \end{cases}$$

Note: Do not write 5

$$f_Y(y) = P(\underbrace{y \le g(x)}_{\text{This is not an event in } \mathcal{F}})$$

~~Rember~~ Remember

$$\{Y \le y\} = \{X \in D_y\} = \underbrace{\{\omega \in \mathcal{S} : X(\omega) \in D_y\}}_{\in \mathcal{F}}$$

Approach 2. (Change of variables)

Assume:

- The function g^{-1} exists, so if $y = g(x)$, then $x = g^{-1}(y)$ for a unique $x \in \mathbb{R}$.
- The function g is differentiable

Then: If $Y = g(X)$,

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\left|\frac{dg(x)}{dx}\right|_{x=g^{-1}(y)}}$$

$$\text{if } \frac{dg(x)}{dx} \neq 0$$

Proof is in Papoulis

It can be shown that if $y = g(x)$ has n solutions $x_1, \dots, x_n$ for finite n, then

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\left|\frac{dg(x)}{dx}\right|_{x=x_i}}$$

Proof in Papoulis

For example, if $g(x) = x^2$, then $x_1 = -\sqrt{y}$, $x_2 = \sqrt{y}$, for $y > 0$.

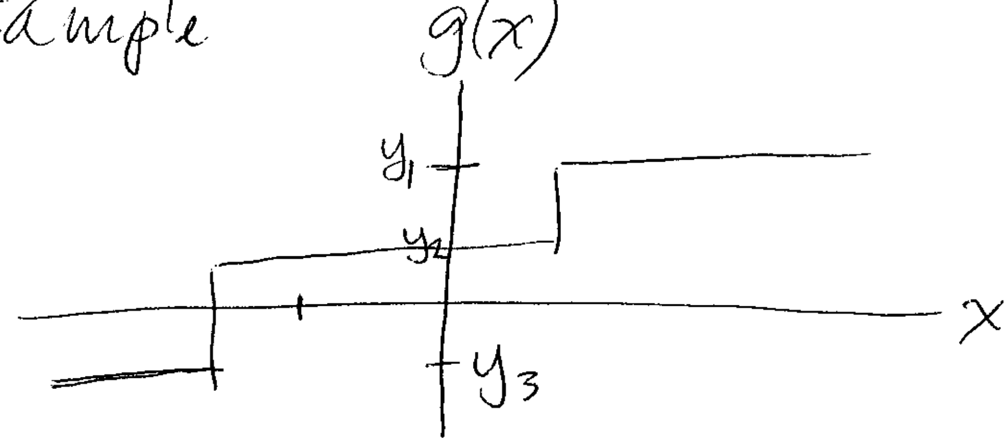
Now case 2: X continuous, Y discrete

To find the pmf of Y,

first identify R_Y.

~~We can let $R_Y = g(\mathbb{R})$, the~~ image of ~~function g~~ set $\mathbb{R}$ under function g.

Example



$$R_Y = \{y_1, y_2, y_3\}$$

Note if $P(Y = y_i) = 0$ for some $y_i \in R_Y$ defined this way, F_Y will not have a jump discontinuity at $y = y_i$. We can keep $y_i \in R_Y$ and use the defn. of a discrete rv as:

Y is discrete if $\exists\, R_Y \subset \mathbb{R}$ such that

- $\mathcal{R}_Y$ is finite or countably infinite, and

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- $P(Y \in \mathcal{R}_Y) = 1$