Conditional Distributions 9/21/2022

How do we represent conditional probs. for rvs?

Recall that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0$$

$\forall A, B \in \mathcal{F}$.

Now consider $A = \{X \leq x\}$ for $x \in \mathbb{R}$.

Then define

$$F_X(x|B) = P(A|B) = \frac{P(\{X \leq x\} \cap B)}{P(B)}$$

This is the <u>conditional cdf of X given B</u>. The conditional pdf of X given B is ~~the~~ a function $f_X(x|B)$ satisfying

$$P(A|B) = \int_A f_X(x|B)\,dx, \quad \text{or}$$

$$f_X(x|B) = \frac{d}{dx} F_X(x|B) \quad \forall x \in \mathbb{R}$$

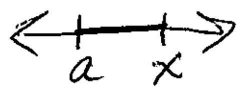
where $F_X(x|B)$ is differentiable.

Example. Let $B = \{X > a\}$ for some $a \in \mathbb{R}$. Assume $P(B) > 0$.

Then $F_X(x|B) = \dfrac{P(\overbrace{X \le x, X > a}^{\{X \le x\} \cap \{X > a\}})}{P(X > a)}$

To write this in terms of f_X and a, first consider two cases:

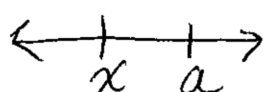
(i) $x > a$



$$F_X(x|X>a) = \frac{P(a < X \le x)}{P(X>a)}$$

$$= \frac{F_X(x) - F_X(a)}{1 - F_X(a)}$$

(ii) $x \le a$

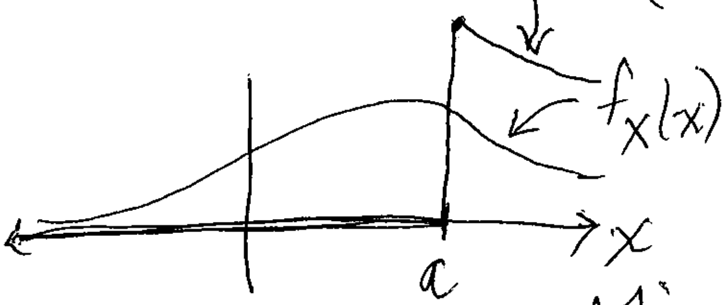


$$F_X(x|X>a) = \frac{P(a < X \le x)}{P(X>a)} =$$

$$\frac{P(\emptyset)}{1 - F_X(a)} = 0$$

So $$F_X(x \mid X > a) = \begin{cases} \dfrac{F_X(x) - F_X(a)}{1 - F_X(a)} & x > a \\ 0 & x \le a \end{cases} \quad (3)$$

Then we have
$$f_X(x \mid X > a) = \begin{cases} \dfrac{f_X(x)}{1 - F_X(a)} & x > a \\ 0 & x \le a \end{cases}$$



This conditional distribution is sometimes called a peaks-over-threshold (POT) distribution, and can be used to model rare events

There is a special forms of Bayes' Theorem for cond. distribution. We have
$$F_X(x \mid B) = \frac{P(\{X \le x\} \cap B)}{P(B)}$$

$$= \frac{P(B \mid X \le x)\, P(X \le x)}{P(B)}$$ ④

assuming $P(X \le x) > 0$

So
$$\boxed{F_X(x \mid B) = \frac{P(B \mid X \le x)\, F_X(x)}{P(B)}}$$

$\forall x$ with $P(X \le x) > 0$, and $P(B) > 0$.

Also, if $A_1, \ldots, A_n$ form a partition of S then

$$F_X(x) = \sum_{i=1}^{n} F_X(x \mid A_i)\, P(A_i)$$

if $P(A_i) \neq 0$ for $i = 1, \ldots, n$

(TPL)

We often want to condition on $\{X = x\}$ instead of $\{X \le x\}$, but if X is continuous we cannot use our original defn of cond. prob since $P(X=x)=0$.

Instead we let

$$P(A|X=x) = \lim_{\Delta x \to 0} P(A \,|\, x < X \leq x + \Delta x)$$

assuming $P(x < X \leq x+\Delta x) > 0$ for all $\Delta x < \varepsilon > 0$

Then

$$P(A|X=x) = \lim_{\Delta x \to 0} \frac{P(x < X \leq x+\Delta x | A) P(A)}{P(x < X \leq x+\Delta x)}$$

$$= P(A) \frac{\lim_{\Delta x \to 0} \frac{1}{\Delta x} P(x < X \leq x+\Delta x | A)}{\lim_{\Delta x \to 0} \frac{1}{\Delta x} P(x < X \leq x+\Delta x)}$$

$$= P(A) \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x}\left[F_X(x+\Delta x|A) - F_X(x|A)\right]}{\frac{1}{\Delta x}\left[F_X(x+\Delta x) - F_X(x)\right]}$$

$$= \frac{f_X(x|A) P(A)}{f_X(x)} \qquad \forall x \text{ where } f_X(x) > 0$$

So
$$\boxed{P(A \mid X=x) = \frac{f_X(x \mid A)\,P(A)}{f_X(x)}}$$

Form of Bayes' Theorem used when X is continuous

When X is discrete can use original form of Bayes' Theorem

$$\underbrace{p_X(x \mid B)}_{} = \frac{P(B \mid X=x)\,p_X(x)}{P(B)}$$

$$\forall x \in R_X$$

the conditional pmf of X given B

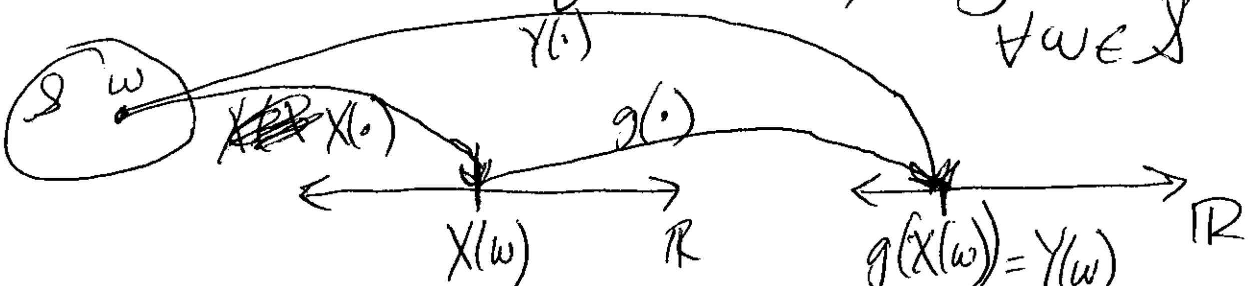
* End of Exam I material

Functions of a random variable

Consider a rv X and a function $g: \mathbb{R} \to \mathbb{R}$. If we know the distribution of X

and the form of g,
can we find the distr. of
$Y = g(X)$?

More precisely $Y(\omega) = g(X(\omega))$ $\forall \omega \in \mathcal{S}$



We will assume that
g satisfies

$$Y^{-1}(A) = \{\omega \in \mathcal{S} : Y(\omega) \in A\} =$$
$$\{\omega \in \mathcal{S} : g(X(\omega)) \in A\} \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathbb{R})$$

so Y is a valid rv,
ie. $P(Y \in A)$ exists $\forall A \in \mathcal{B}(\mathbb{R})$.