Properties of the pmf: 9/22/2022

① $p_X(x) \geq 0 \quad \forall x \in \mathcal{R}_X$

② $\sum_{x \in \mathcal{R}_X} p_X(x) = 1$

③ $P(X \in A) = \sum_{x \in A} p_X(x)$ for any $A \subset \mathcal{R}_X$

Note: It can be shown that a function satisfying Properties ① and ② is a valid pmf

Ex. Let X be the sum of values on two die rolls

$\mathcal{R}_X = \{2, 3, \ldots, 12\}$

and if the die is fair, then

$$p_X(x) = \begin{cases} \frac{1}{36} & x = 2, 12 \\ \vdots & \vdots \\ \frac{6}{36} & x = 7 \end{cases}$$

Use property 3 with A = {two rolls sum to 7}

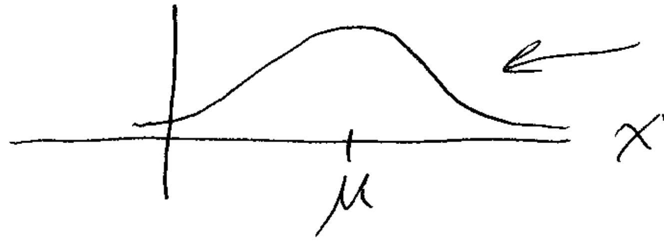
Some important (named) rvs

1. Gaussian or Normal

Continuous

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \forall x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\sigma > 0$



symmetric about μ

No closed form solution for

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

for Gaussian X. Tables can be used to find probs in this case.

Notation: $X \sim N(\mu, \sigma^2)$

Sometimes used to model:

- noise
- sums of rvs.

Central Limit Theorem

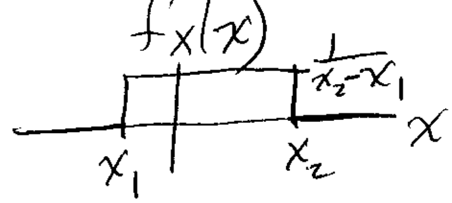
- Continuous rvs whose histograms look like a bell curve

(2) Uniform

(i) Continuous case

$$f_X(x) = \begin{cases} \frac{1}{x_2 - x_1} & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases}$$

for some $x_1, x \in \mathbb{R}$ with $x_2 > x_1$



(ii) discrete case

$R_X = \{x_1, \dots, x_n\}$ for some finite ~~$n \neq 0$~~ integer $n > 0$, $x_i \in \mathbb{R}$ $\forall i = 1, \dots, n$

$$p_X(x_i) = \frac{1}{n} \quad \forall x_i \in R_X$$

Used when outcomes are equally likely.
For the continuous case this means that

$$P(X \in [a,b]) = \frac{|b-a|}{|x_2 - x_1|}$$

for any $a, b \in [x_1, x_2]$ with $b > a$

(3) Binomial

Discrete $R_X = \{0, \ldots, n\}$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n$$

$0 \le p \le 1$, $n \ge 1$, n finite

X can be viewed as the number of successes in n ~~trial~~ Bernoulli trials with P(~~suc~~ success) = p in each trial

Proof: View the choice of the k successes in n trials as a combinatorics problem

$\underline{A^c}\ \underline{A^c}\ \underline{A}\ \underline{A}\ \ldots\ \underline{A}\ \underline{A^c}$ n slots

Pick k slots from n to put successes.

- A slot cannot be chosen twice (no replacement)
- It does not matter which order the slots for successes are chosen

So from combin., there are

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose k successes in n trials

Now

$$\{k \text{ succ. in } n \text{ trials}\} = \bigcup_{i=1}^{\binom{n}{k}} \underbrace{\{i\text{th comb. occurs}\}}_{\substack{\text{elementary event} \\ \text{(an event containing} \\ \text{exactly one outcome)}}}$$

using some arbitrary ordering of the $\binom{n}{k}$ combos.

The elementary events are disjoint, so

$P(k$ ~~succs~~ successes in n trials$) =$

$$\sum_{i=1}^{\binom{n}{k}} P(i\text{th comb. occurs})$$

But $P(i\text{th comb}) = p^k(1-p)^{n-k}$

since Bernoulli trials are independent. So

$P(k$ succ. in n trials$) =$

$$\sum_{i=1}^{\binom{n}{k}} p^k(1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

4 Geometric

Discrete

Form 1: $p_X(k) = p(1-p)^k, \; k = 0, 1, \ldots$

Form 2: $p_X(k) = p(1-p)^{k-1}, \; k = 1, 2, \ldots$

$0 \le p \le 1$

Used to model the number of Bernoulli trials before first success

5 Poisson

Discrete

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\lambda \in \mathbb{R}$, $\lambda > 0$

This is the limiting function when $n \to \infty$, $np = \lambda$ for the ~~Bernoulli~~ Binomial rv.

Used to model the number of occurrences of an event in a given interval of time, space, etc.

- Customers arriving at a store
- packets arriving at a given router
- particles hitting a detector
- low-particle-count noise