Combined Experiments

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Consider two experiments $(S_1, \mathcal{F}_1, P_1)$ and $(S_2, \mathcal{F}_2, P_2)$. To model these exper. jointly, we need a combined prob space $(S, \mathcal{F}, P)$.

Let $S = S_1 \times S_2 = \{(\omega_1, \omega_2) : \omega_1 \in S_1, \omega_2 \in S_2\}$

↑ Cartesian product

If S is not uncountable, let

$$\mathcal{F} = \mathcal{P}(S)$$

If $S_1 = S_2 = \mathbb{R}$, then let

$$\mathcal{F} = B(\mathbb{R}^2) = \sigma(\{\text{open rectangles in } \mathbb{R}^2\}),$$

where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

This is called $B(\mathbb{R}^2)$, the Borel field on $\mathbb{R}^2$

Now we need $P(C)$ $\forall C \in \mathcal{F}$.

First, let $P(A \times S_2) = P_1(A)$ and

$P(S_1 \times B) = P_2(B)$, $\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2$

Now consider events of the type $A \times B$ for $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$. Let

$$P(A \times B) = P((A \times S_2) \cap (S_1 \times B)) =$$

$$P(A \times S_2) P(S_1 \times B) = P_1(A) P_2(B)$$

equality by assumption

If $C \in \mathcal{F}$ and $C \neq A \times B$ for any ~~A,~~ $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$, ~~then~~ the axioms can be used to find $P(C)$

Example where $C \neq A \times B$, $C \in \mathcal{F}$:

Let $S_1 = \{H, T\}$, $S_2 = \{1, 2, 3, 4, 5, 6\}$

Then, for example,

$C = \{(H,2), (T,6)\}$ is not of the form $A \times B$, but is $\in \mathcal{F}$

Let $A = \{H\}$, $B = \{1, 2\}$, then $A \times B = \{(H,1), (H,2)\}$

Bernoulli Trials

A special case of combined experiments that models repeated independent trials of an experiment where one particular event is of interest in the trials. Consider an experiment $(\mathcal{S}_0, \mathcal{F}_0, P_0)$ and event $A \in \mathcal{F}_0$, with $P_0(A) = p$, $0 \le p \le 1$.

To model n independent trials, let

$$\mathcal{S} = \underbrace{\mathcal{S}_0 \times \cdots \times \mathcal{S}_0}_{n \text{ terms}}$$

$$= \{(\omega_1, \ldots, \omega_n) : \omega_i \in \mathcal{S}_0 \ \forall i\}$$

For ex. if $\mathcal{S}_0 = \{H, T\}$, and $n = 3$, then (H, T, H) would be one of the outcomes in $\mathcal{S}$.

Two common questions:

1. What is the prob. of getting

~~k~~ k successes (or occurrences of A) in n trials, for some $k \in \{0, \ldots, n\}$?

2. What is the prob. of k trials before the first success?

We will address these when we discuss the binomial and geometric random variables.

Random Variables

As engineers we typically work with measured, or sensed, data, in the form of variables. Since these measurements are random, in general, we use random variables.

A rv X defined on a space $(\mathcal{S}, \mathcal{F}, P)$ is a function $X : \mathcal{S} \to \mathbb{R}$.

We will need to restrict the set

of real-valued functions on $\mathcal{S}$ that can be r.v.s.

Defn. Given two spaces $\mathcal{S}$ and $\mathcal{R}$, an $\mathcal{R}$-valued function $f : \mathcal{S} \to \mathcal{R}$ assigns a value from $\mathcal{R}$ to every element of $\mathcal{S}$

$$f(\omega) \in \mathcal{R} \quad \forall \omega \in \mathcal{S}.$$

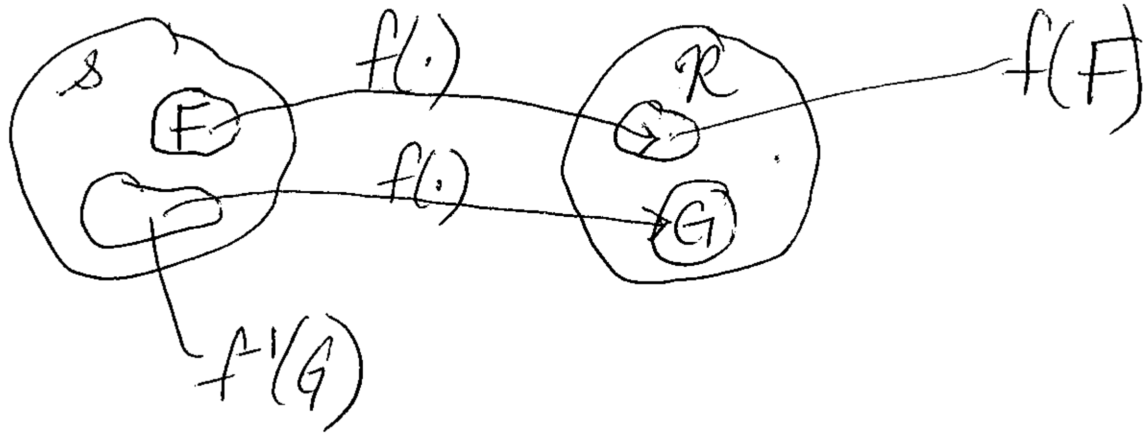
Defn. Given any $F \subset \mathcal{S}$ and $G \subset \mathcal{R}$, the ~~inverse of~~ image of F under f is

$$f(F) = \{ a \in \mathcal{R} : a = f(\omega) \text{ for some } \omega \in F \}$$

and the inverse image of G under f is

$$f^{-1}(G) = \{ \omega \in \mathcal{S} : f(\omega) \in G \}$$

for example



Note: The inverse image $f^{-1}(G)$ is not the same as the inverse function $f^{-1}(a)$, $a \in \mathcal{R}$.

$f^{-1}(G)$ exists $\forall G \subset \mathcal{R}$

$f^{-1}(a)$ may not be well-defined for $a \in \mathcal{R}$.