HW1 and HW2 will be posted today. 9/1/2022

HW1 can be turned in Thurs. 9/8 instead of Tue. 9/6.

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The Event Space $\mathcal{F}$ ~~(cont'd~~ (cont'd)

$\mathcal{F}$ is the set of all events

If $A$ is an event, it is in $\mathcal{S}$ (a subset of $\mathcal{S}$)

If ~~is a sub~~ $A \subset \mathcal{S}$, it may or may not be an event

If $A \in \mathcal{F}$ (or $A$ is an event) $P(A)$ is defined

If $A \notin \mathcal{F}$, $P(A)$ is not defined

Can an arbitrary set of subsets of $\mathcal{S}$ be an event space?

No.

A valid event space must be <u>non-empty</u>, and satisfy

1. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

   (Note: $A \subset S$, but $A \in \mathcal{F}$)

2. For any finite $n \geq 1$ and any $A_1, \ldots, A_n \in \mathcal{F}$,

   $$\bigcup_{i=1}^{n} A_i \in \mathcal{F}$$

3. For any sequence $A_1, A_2, \ldots \in \mathcal{F}$,

   $$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

A set of sets satisfying these properties is called a <u>$\sigma$-field</u> in math

(Aside: For a universal set $S$,

$w \in S$ means $w$ is an element of $S$

$A \subset S$ means $A$ is a set containing elements of $S$

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$A \in \mathcal{F}$ means $A$ is an event

$A \notin \mathcal{F}$ means $A$ is not an event )

Example. For $S = \{H, T\}$

$\mathcal{F} = \{\emptyset, \{H, T\}\}$ is a valid event space

$\mathcal{F} = \{\emptyset, \{H\}, \{T\}, S\}$ is also valid

But $\mathcal{F} = \{\emptyset, \{H\}, S\}$ is not valid

Also $\mathcal{F} = \emptyset$ is not valid.

$\mathcal{F} = S$ is not valid, because this $\mathcal{F}$ is not a set of subsets of $S$

$\mathcal{F} = \{S\}$ is not valid,

because $S^c = \phi$ is not in $\mathcal{F}$

So the event space must be a $\sigma$-field.

Some comments

- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$

Proof.

$$A \cap B = (A^c \cup B^c)^c$$

By property 1, $A^c, B^c \in \mathcal{F}$

By property 2 $A^c \cup B^c \in \mathcal{F}$

by prop. 1 again $\underbrace{(A^c \cup B^c)^c}_{A \cap B} \in \mathcal{F}$

- For any $S, \mathcal{F}$, $\phi \in \mathcal{F}$, $S \in \mathcal{F}$

Proof omitted.

- For any $S$, $\mathcal{P}(S)$ (the set of all subsets of $S$) is a valid $\sigma$-field. If $S$ is finite or countable, $\mathcal{P}(S)$ is a valid event space. If $S = \mathbb{R}$, $\mathcal{P}(S)$ is a $\sigma$-field but will not be allowed as the event space (more on that later)

- For a finite $S$, Property 2 of $\sigma$-field defn. $\Rightarrow$ Prop. 3 is true

- For a non-finite $S$, Property 3 $\Rightarrow$ Prop. 2 is true

If $S$ is finite or countable, can assume $\mathcal{F} = \mathcal{P}(S)$ unless you are told otherwise.

$\mathcal{F}$ for $S = \mathbb{R}$

It can be shown that in ~~one~~ order for the axioms of prob. to be usable, some subsets of $\mathbb{R}$ cannot be events. (Proof outside the scope of this class).

So $\mathcal{P}(\mathbb{R})$ cannot be $\mathcal{F}$, even though it is a $\sigma$-field.

We will construct $\mathcal{F}$ for the case $S = \mathbb{R}$ now, and use that $\mathcal{F}$ in this class.

Defn. Given a space $S$, consider a set of subsets

$G = \{A_i \subset S,\ i \in I\}$ ⑦

Then the $\sigma$-field generated by $G$, denoted $\sigma(G)$, is the smallest $\sigma$-field that contains every set in $G$, or

$$\sigma(G) = \bigcap_{i \in I_G} \mathcal{F}_i$$

where $\{\mathcal{F}_i,\ i \in I_G\}$ is the set of all $\sigma$-fields containing $G$.

If $G$ is a $\sigma$-field, then

$$\sigma(G) = G$$

Now, when $S = \mathbb{R}$, we will let

$$\mathcal{F} = \sigma(G) \text{ with}$$

$$G = \{(a,b) \subset \mathbb{R} : a < b\}$$

set of all open intervals in $\mathbb{R}$

This $\mathcal{G}$ is not a $\sigma$-field. For example

$$(a,b)^c = \underbrace{(-\infty, a] \cup [b, \infty)}_{\text{not an open interval.}}$$

This $\sigma(\mathcal{G})$ is called the <u>Borel field on $\mathbb{R}$</u>, denoted $\mathcal{B}(\mathbb{R})$.

Some sets in $\mathcal{B}(\mathbb{R})$:

- $S = (-\infty, \infty) = \bigcup_{n=1}^{\infty} (-n, n) \in \mathcal{B}(\mathbb{R})$
- $\phi = S^c \in \mathcal{B}(\mathbb{R})$
- $(-\infty, b) = \bigcup_{i=1}^{\infty} (-n, b) \in \mathcal{B}(\mathbb{R})$

  (let $(-n, b) = \phi$ if $b \leq -n$)
- $\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \in \mathcal{B}(\mathbb{R})$