Set theory (cont'd)

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Defn. Sets A and B are <u>disjoint</u> if they contain no common elements, or if

$$\omega \in A \Rightarrow \omega \notin B \text{ and}$$
$$\omega \in B \Rightarrow \omega \notin A$$

(In prob. this might also be called "mutually exclusive")

Notation:

$\Rightarrow$ means implies or implies that

Note that $\longrightarrow$ does not have a clear meaning

Defn. The <u>complement</u> of a set A is

$$A^c = \{\omega \in \mathcal{S} : \omega \notin A\}$$

Also denoted $\overline{A}$

Defn. The <u>union</u> of sets A and B is

$$A \cup B = \{\omega \in \mathcal{S} : \omega \in A \text{ or } \omega \in B \text{ or both}\}$$

Defn. The ~~co~~ intersection of sets A and B is

$$A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$$

Set Algebra

The operations $\cap$, $\cup$, and complement follow certain rules of algebra.

Two sets are equal if they contain exactly the same elements.

Set algebra is concerned with equality of sets.

One example. For any space S,

$$S \cup \phi = S$$

To prove this:
Let $\omega \in S \cup \phi$. Then
$\omega \in S$ or $\omega \in \phi$ or both, from defn. of $\cup$.
This means $\omega \in S$, since ϕ has no elements

Conversely, let $w \in S$.
Then $w \in \{a \in S : a \in S$ or $a \in \emptyset$ or both$\}$
since $w \in S$. Thus
$w \in S \cup \emptyset$

QED

In general, can show that $A = B$ using:
(1) Let $w \in A$ and show $w \in B$, and
(2) Let $w \in B$ and show $w \in A$.

Defn. A set A is a <u>subset</u> of set B if

$$w \in A \Rightarrow w \in B.$$

Notation: $A \subset B$

Note: Sometimes '$\subset$' is used for a proper subset, which means $A \subset B$ but B contains elements not in A, so $A \neq B$.

In this class,
$A \subset B$ will also be used for the case $A = B$.

Note that in this class, any set is a subset of itself.
Also $A = B$ iff $A \subset B$ and $B \subset A$.

A document on set theory will be posted. It will have many properties from set algebra. Some important properties:

- $\cup, \cap$ are
  - commutative
    $A \cup B = B \cup A$
  - associative
    $(A \cup B) \cup C = A \cup (B \cup C)$
- $\cap$ distributes over $\cup$ and $\cup$ distr. over $\cap$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proofs are omitted.

Three types of sets:

1. A set is <u>finite</u> if it has a finite number of elements $A = \{x_1, \dots, x_n\}$ for some finite n.

2. A set is <u>countably infinite</u>, or <u>countable</u>, if its elements can be put into one-to-one correspondence with the integers. So the set of positive integers is countable:

Integers		Positive Integers
0	$\longleftrightarrow$	1
−1	$\longleftrightarrow$	2
1	$\longleftrightarrow$	3
−2	$\longleftrightarrow$	4
2	$\longleftrightarrow$	5
⋮		⋮

Note: A finite set is not countable.

Note: In this class, countable and countably infinite mean the same thing. Other sources may use countable to mean either finite or countably infinite.

A countable set A can be written as

$$A = \{x_1, x_2, \ldots\}$$

or

$$A = \{x_0, x_1, \ldots\}$$

Also, it can be shown that the rational numbers $\mathbb{Q}$ form a countable set.

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \right\}$$