The recursive form of the linear MMSE filter for estimating Z_n from $X_1, \ldots, X_{n-1}$, with initialization Z_0, $X_0 = \cancel{Z_0}$: Dec 8, 2022

Non-recursive form:

$$R_{ZX}(n,l) = \sum_{j=1}^{n} h_j^{(n-1)} R_{XX}(n-j,l),$$
$$l = 0, \ldots, n-1.$$

Recall
$$Y_n = \sum_{j=1}^{n} h_j^{(n-1)} X_{n-j}$$

Will not derive recursive form, but start with some observations:

- $R_{ZX}(n+1, l) = E[\underbrace{(a_n Z_n + W_n)}_{Z_{n+1}\text{, from state model}} X_l]$

$$= a_n E[Z_n X_\ell] + E[W_n X_\ell], \quad \ell = 0, \dots, n-1$$

W_n and X_ℓ are uncorrelated, since the model error W_n is uncorrelated with all previous observations $X_0, \dots, X_{n-1}$

So $$R_{ZX}(n+1, \ell) = a_n E[Z_n X_\ell] + \underbrace{E[W_n]}_{=0} E[X_\ell]$$

$$= a_n R_{ZX}(n, \ell)$$

- $$R_{ZX}(n, \ell) = E[\underbrace{(X_n - N_n)}_{Z_n \text{ from observation model}} X_\ell]$$

$$= E[X_n X_\ell] - \underbrace{E[N_n X_\ell]}_{=0}$$

$$= R_{XX}(n, \ell)$$

So $$R_{ZX}(n+1, \ell) = a_n R_{XX}(n, \ell)$$

Using these observations, (3)
the following system can
be derived:

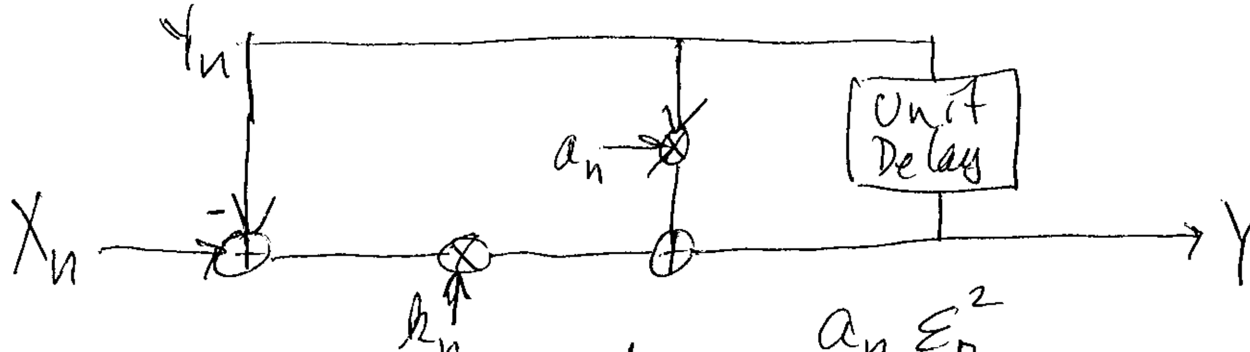
$$Y_{n+1} = a_n Y_n + k_n \underbrace{(X_n - Y_n)}$$

estimate of Z_{n+1}

measurement at t_n − estimate at t_n

- $X_n - Y_n$ is called the "innovations sequence"
- k_n is the "gain" of the filter

$$k_n = \frac{a_n E[(Z_n - Y_n)^2]}{E[(Z_n - Y_n)^2] + E[N_n^2]}$$



$$k_n = \frac{a_n \varepsilon_n^2}{\varepsilon_n^2 + E[N_n^2]}$$

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Note that $0 < a_n <$. If $\dfrac{a_n \varepsilon_n^2}{\varepsilon_n^2 + E[N_n^2]} \approx 0$,

gain is ≈ 0

If $\dfrac{a_n \varepsilon_n^2}{\varepsilon_n^2 + E[N_n^2]} \approx a_n$,

gain is close to its highest value

The KF is very widely used, but ~~has~~ the linear models

$$Z_n = a_n Z_{n-1} + W_n$$

$$X_n = Z_n + N_n$$

limit its use.

The particle filter uses much more general models.

Particle Filter Overview

State model:

$$Z_n = g(Z_{n-1}) + W_{n-1}$$

Observation model

$$X_n = h(Z_n) + N_n$$

for functions g, h

The PF seeks the value z_n that maximizes

$$p(z_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1})$$

The maximum a posteriori (MAP) estimate of Z_n given $X_1, \ldots, X_{n-1}$ and initialization $X_0 = x_0$

At each ~~st~~ time t_{n+1}:

Update step:

$$p(z_n | x_0, \ldots, x_n) = \frac{p(x_n | z_n)\, p(z_n | x_0, \ldots, x_{n-1})}{\int p(x_n | z_n)\, p(z_n | x_0, \ldots, x_{n-1})}$$

Note that

- The term $p(z_n|x_0, \dots, x_{n-1})$ was estimated ~~to~~ at the previous time
- The denominator is from the TPL

Prediction Step

$$p(z_{n+1}|x_0, \dots, x_n) = \int p(z_{n+1}|z_n)\, p(z_n|x_0, \dots, x_n)\, dz_n$$

You do not need to know where the expressions in the update and prediction steps come from.