Dec 1, 2022

Have shown that if $X(t)$ is a WSS input to an LTI system with impulse response $h(t)$, then the output $Y(t)$ has

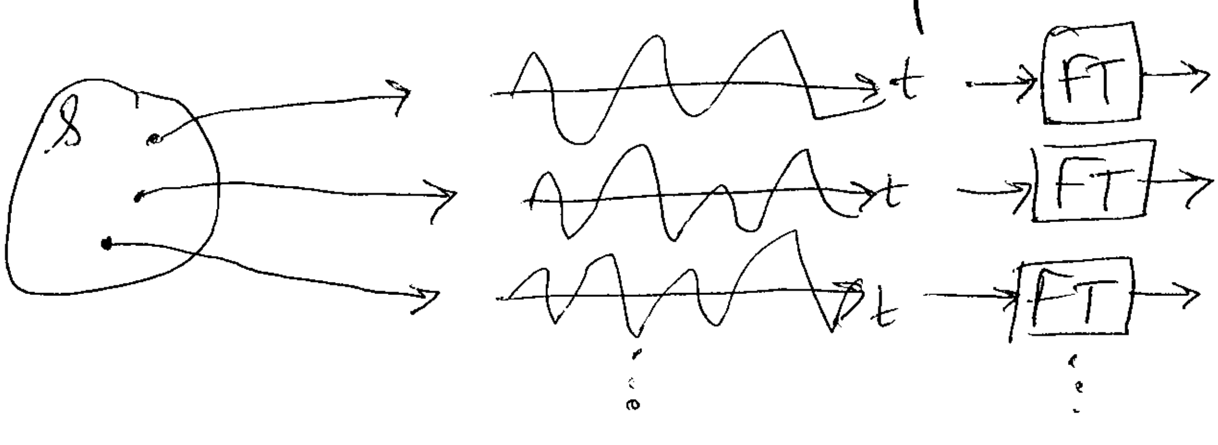
- $\mu_Y(t) = \mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\,dt$
- $R_{YY}(t_1,t_2) = R_Y(\tau) = (R_X * h * \tilde{h})(\tau)$,

  $\tau = t_2 - t_1$, $\tilde{h}(t) = h(-t)$

So $Y(t)$ is WSS, if $\int_{-\infty}^{\infty} h(t)\,dt < \infty$

The Power Spectrum of a rp

What is the "frequency content" of a rp



Taking a ~~FT of~~ Fourier Transform of each sample realization of the rp is not very informative about the overall rp. Instead, use:

Defn. The <u>power spectral density</u> (PSD) of a WSS rp $X(t)$ is the Fourier transform of its autocorrelation function:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau, \quad \omega \in \mathbb{R}$$

Some properties of S_X: (assuming that $X(t)$ is real-valued, not complex)

① Since $R_X(-\tau) = R_X(\tau)$ (e.g., R_X is an even function), $S_X(\omega)$ is real $\forall \omega \in \mathbb{R}$

② $R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{i\omega\tau} d\omega$

③ $R_X(\tau)$ real $\Longrightarrow S_X(\omega)$ is even

Defn. A WSS rp $X(t)$ is a white-noise process if $\mu_X(t) = 0 \ \forall t$ and $R_X(\tau) = N_0 \delta(\tau)$ for some "noise power" N_0. This means that for white noise,

$$S_X(\omega) = N_0 \quad \forall \omega \in \mathbb{R}$$

Recall that for an LTI system, if $X(t)$ is WSS, then

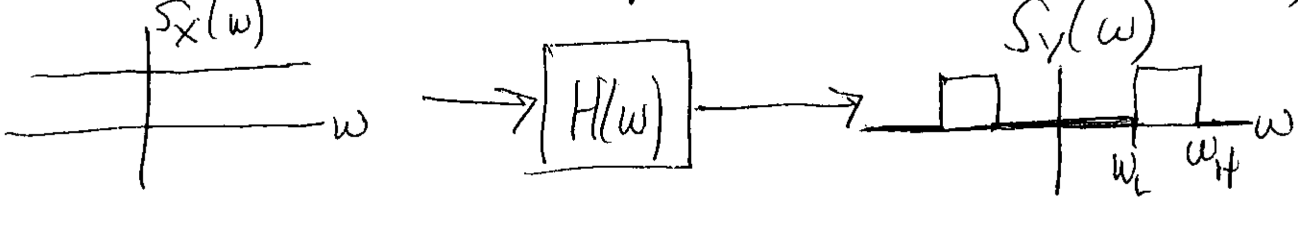
$$R_Y(\tau) = (R_X * h * \tilde{h})(\tau), \quad \text{so}$$

$$S_Y(\omega) = S_X(\omega) H(\omega) H^*(\omega)$$

$$= |H(\omega)|^2 S_X(\omega)$$

where $H(\omega) = \mathcal{F}\{h(t)\}$

Consider white noise $X(t)$, and a bandpass filter $H(\omega)$



where ω_L, ω_H are cutoff frequencies of the bandpass filter. This is how colored noise is created

Defn. The <u>cross-correlation function</u> of two rps $X(t)$ and $Y(t)$ is

$$\tilde{R}_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

Defn. The rps $X(t)$ and $Y(t)$ are <u>jointly WSS</u> if each is WSS and

$$\tilde{R}_{XY}(t_1, t_2) = R_{XY}(\tau)$$

for some $R_{XY} : \mathbb{R} \to \mathbb{R}$, $\tau = t_2 - t_1$

Defn. The <u>cross power spectral density</u> of jointly WSS rps $X(t)$ and $Y(t)$ is

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

It can be shown that

if $\underset{\text{WSS}}{X(t)} \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$

then $R_{XY}(\tau) = (h * R_X)(\tau)$

which means $S_{XY}(\omega) = H(\omega) S_X(\omega)$

Example. If $X(t)$ is white noise, and $h(t) = e^{-t}u(t)$, find μ_Y, R_Y, R_{XY}.

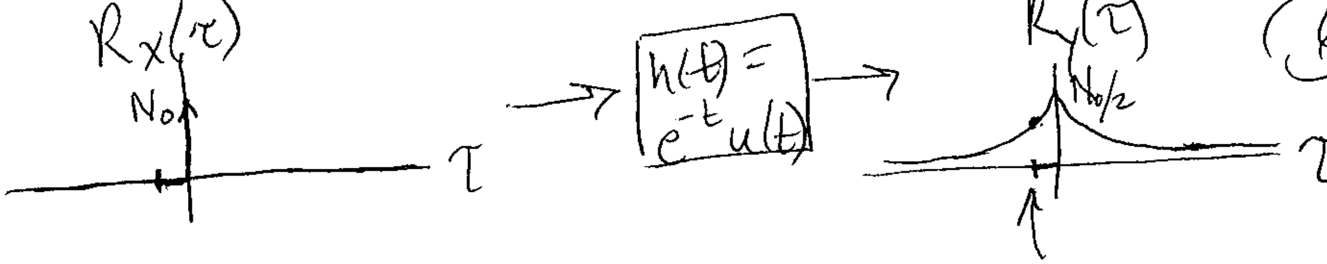
$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\,dt = 0, \text{ since } \mu_X (0)$$

To find R_Y using the freq. domain:

$$H(\omega) = \frac{1}{1 + i\omega}$$

$$S_Y(\omega) = S_X(\omega) |H(\omega)|^2 = \frac{N_0}{1+\omega^2}$$

So $R_Y(\tau) = \frac{N_0}{2} e^{-|\tau|}$



Could also solve in the time domain — this task is left to the student

Finding R_{XY} is left to the student