11/8/2022

Exam 2

- No mgf/char. fn. questions
- Explain your work!
- Integrate constants, polynomials, e^{ax}; leave more complicated integrals

Notation for convergence in distribution: $X_n \xrightarrow{d} X$

Note: It is possible for a sequence $X_1, X_2, \ldots$ to converge in distr., but to have density fns that do not converge $\forall x \in \mathbb{R}$.

Example. Consider a sequence of rvs $X_1, X_2, \ldots$ with

$$f_{X_n}(x) = 1 + \cos 2\pi n x, \quad 0 \le x \le 1,$$

$n = 1, 2, \ldots$; and 0 elsewhere

Then $F_{X_n}(x) \longrightarrow \begin{cases} 0 & x<0 \\ x & 0 \le x \le 1 \\ 1 & x>1 \end{cases}$

as $n \to \infty$

So X_n converges in distr. to a unif $[0,1]$ rv

But f_{X_n} does not converge except at $x=0$, $x=1$

Relationships among ~~types~~ of types of convergence:

$$\begin{tikzcd} e \arrow[r, Rightarrow] & ae \arrow[r, Rightarrow] & p \arrow[r, Rightarrow] & d \\ & ms \arrow[ur, Rightarrow] & & \end{tikzcd}$$

Proof outside scope of class

<u>The Cauchy Criterion for MS convergence</u>

It is possible to show MS convergence without knowing the limiting rv X. First consider a sequence $x_1, x_2, \ldots$, where $x_i \in \mathbb{R}$ $\forall i$. The Cauchy criterion states that if $|x_{n+m} - x_n| \to 0$ as $n \to \infty$, for any $m \geq 1$, then x_n converges to some limit x. The converse is also true.

Proof omitted. (from real analysis)

For a sequence of rvs $X_1, X_2, \ldots$, the Cauchy criterion for convergence states that $X_1, X_2, \ldots$ converges in MS iff

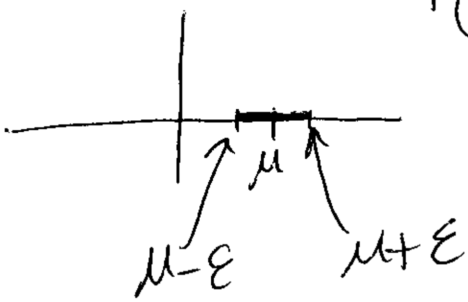
$$E\left[|X_{n+m} - X_n|^2\right] \to 0 \text{ as } n \to \infty, \; \forall m \geq 1.$$

The Chebyshev Inequality

Let X be a rv with finite mean μ and variance σ^2. Then

$\forall \varepsilon > 0$,

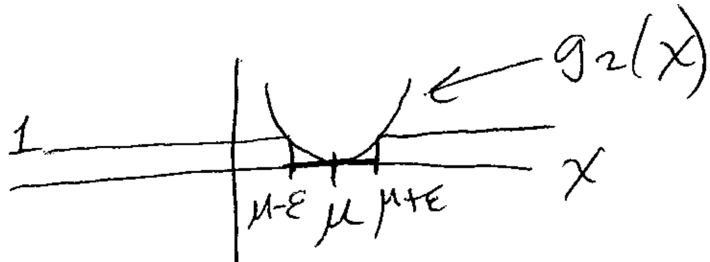
$$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$



Proof. let $g_1(x) = I_{\{x \in \mathbb{R} : |x-\mu| \geq \varepsilon\}}(x)$

and $g_2(x) = \dfrac{(x-\mu)^2}{\varepsilon^2}$

for some arbitrary $\varepsilon > 0$.



Now let $\phi(x) = g_2(x) - g_1(x)$
$\geq 0 \ \forall x$

So $E[\phi(X)] \geq 0$.

But $E[\phi(X)] = E[g_2(X)] - E[g_1(X)]$

$= \dfrac{\sigma^2}{\varepsilon^2} - P(|X-\mu| \geq \varepsilon)$

since $E[g_1(X)] = 0 \cdot P(|X-\mu| < \varepsilon) + 1 \cdot P(|X-\mu| > \varepsilon)$

So $\frac{\sigma^2}{\varepsilon^2} - P(|X-\mu| \geq \varepsilon) \geq 0$,

which means $P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$

The Laws of Large Numbers

Consider a sequence of rvs $X_1, X_2, \ldots$, and the sequence of sample means

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

The Weak Law of Large Numbers (WLLN) states that

$$P(|Y_n - \mu| \geq \varepsilon) \to 0 \quad \text{as} \quad n \to \infty$$

if $X_1, X_2, \ldots$ is an iid sequence with finite mean μ and variance σ^2. In other words

$$Y_n \to \mu \text{ in prob.}$$

Proof. First note that

$$E[Y_n] = \mu, \quad \text{and}$$

$$Var(Y_n) = \frac{\sigma^2}{n}$$

then, from the CI,

$$P(|Y_n - \mu| \geq \varepsilon) \leq \frac{Var(Y_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$$\longrightarrow 0 \text{ as } n \to \infty,$$

$$\forall \varepsilon > 0.$$