Gaussian Random Vectors

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n rvs $X_1, \dots, X_n$ are jointly Gaussian if
$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |C_{\underline{X}}|}} \exp\left[-\frac{1}{2}(\underline{X}-\mu_{\underline{X}})^T C_{\underline{X}}^{-1} (\underline{X}-\mu_{\underline{X}})\right]$$

where $C_{\underline{X}}$ is the covariance matrix of $\underline{X}$, and $\mu_{\underline{X}}$ is the mean vector.

Aside: Note that:

- $C_{\underline{X}}$ is invertible since it is NND
- $C_{\underline{X}}^{-1}$ is NND
- The pdf $f_{\underline{X}}(\underline{x})$ has its maximum at $\underline{X} = \mu_{\underline{X}}$, and the <u>quadratic</u> <u>term</u> is convex
  without the minus sign

End of Aside.

Note that if $X_1, \dots, X_n$ are zero mean and (pairwise) uncorrelated, then $C_{\underline{X}}$ is diagonal and

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^{n} \sigma_i^2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \frac{x_i^2}{\sigma_i^2}\right]$$

The <u>characteristic function</u> of a rV $[X_1, \dots, X_n]^T$ is

$$\Phi_{\underline{X}}(\underline{\Omega}) = E\left[e^{i\sum_{j=1}^{n} \omega_j X_j}\right]$$

where $\underline{\Omega} = [\omega_1, \dots, \omega_n]^T \in \mathbb{R}^n$

It can be shown that if $X_1, \dots, X_n$ are iid rvs with common char. fn. $\Phi_X = \Phi_{X_i}$, $i = 1, \dots, n$, then $\Phi_Z(\omega) = (\Phi_X(\omega))^n \quad \forall \omega \in \mathbb{R}$ if $Z = \sum_{j=1}^{n} X_j$

Final comment on Gaussian RVs: It can be shown that $X_1, \ldots, X_n$ are jointly Gaussian iff the rv

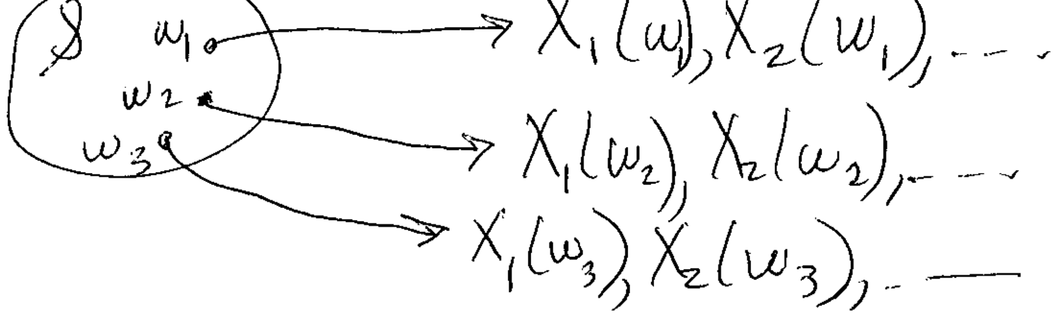
$$Z = a_0 + \sum_{j=1}^{n} a_j X_j$$

is a Gaussian rv $\forall$ vector $[a_0, \ldots, a_n]^T \in \mathbb{R}^{n+1}$

Proof: Omitted; can be proved using $\Phi_{\underline{X}}(\underline{\Omega})$

Stochastic Convergence

Consider an infinite sequence of rvs $X_1, X_2, \ldots$, or $X_1(\omega), X_2(\omega), \ldots$ for completeness



Before defining convergence of a sequence of rvs, recall the case of real-valued sequences:

Defn. A sequence of real numbers $x_1, x_2, \ldots$ converges to the limit $x \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ such that

$$|x - x_n| < \varepsilon \quad \forall n \geq n_\varepsilon .$$

If such an x exists, we write

$$\lim_{n \to \infty} x_n = x, \quad \text{or}$$

$$x_n \to x \quad \text{as } n \to \infty$$

For rvs $X_1, X_2, \ldots$, convergence is more complicated.

Types of Convergence

Defn. X_n <u>converges everywhere</u>, or surely, if the sequence of real numbers $X_1(\omega), X_2(\omega), \ldots$

converges to a real number $X(\omega)$ for each fixed $\omega \in S$. Note that the limit $X(\omega)$ depends on ω in general, so the limit is a rv.

We write

$$X_n \xrightarrow{e} X, \quad \text{or}$$

$$X_n \longrightarrow X \text{ everywhere}$$

This is a very strict type of convergence and is not usually useful in practice

Defn. X_n converges almost everywhere, or almost surely, if

$$X_n(\omega) \to X(\omega) \quad \forall \omega \in A \text{ for some}$$
$$A \in \mathcal{F} \text{ with } P(A) = 1.$$

Also called with ~~po~~ prob. 1

We write $X_n \xrightarrow{ae} X$ or $X_n \xrightarrow{wp1} X$, or $X_n \to X$ wp1, ...

Defn. X_n <u>converges in mean-square</u> to X if $E[|X_n - X|^2] \to 0$ as $n \to \infty$

Note that

$$E[|X_n - X|^2] = \iint_{\mathbb{R}^2} |x_n - x|^2 f_{X_n X}(x_n, x)\, dx_n\, dx$$

is a sequence of real numbers for $n = 1, 2, \ldots$

Write $X_n \xrightarrow{ms} X$

Defn. X_n <u>converges in probability</u> to X if

$$P(|X_n - X| > \varepsilon) \to 0 \quad \text{as } n \to \infty$$

for any $\varepsilon > 0$.

Defn. X_n <u>converges in distribution</u> to X if $F_{X_n}(x) \to F_X(x)$ as $n \to \infty$ $\forall x \in \mathbb{R}$