Linear Time-Invariant Systems With Random Inputs

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Consider the system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

where $x(t)$ is deterministic and $h(t)$ is LTI. Recall that $y(t) = x(t) * h(t)$.

– Now consider a random input $X(t)$.

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

for every $\omega \in S$, $Y(t,\omega) = X(t,\omega) * h(t)$

$$= \int_{-\infty}^{\infty} X(t-\alpha,\omega)\, h(\alpha)\, d\alpha$$

Denote this as $Y(t) = X(t) * h(t)$

What are the mean and autocorrelation functions of $Y(t)$?

First, $E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha\right]$

$$= \int_{-\infty}^{\infty} E[X(t-\alpha)]\, h(\alpha)d\alpha$$

$$= \int_{-\infty}^{\infty} \mu_X(t-\alpha)\, h(\alpha)\, d\alpha = \mu_X(t) * h(t)$$

Now R_{YY}:

$$R_{YY}(t_1,t_2) = E[Y(t_1)\,Y(t_2)]$$

$$= E\left[\int_{-\infty}^{\infty} X(t_1-\alpha)h(\alpha)d\alpha \int_{-\infty}^{\infty} X(t_2-\beta)h(\beta)d\beta\right]$$

$$= \iint_{\mathbb{R}^2} E[X(t_1-\alpha)X(t_2-\beta)]\, h(\alpha)h(\beta)\, d\alpha\, d\beta$$

So $$R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} R_{XX}(t_1-\alpha, t_2-\beta)h(\alpha)h(\beta)\, d\alpha\, d\beta$$

Now consider the case where $X(t)$ is WSS:

$$\mu_Y(t) = \int_{-\infty}^{\infty} \mu_X(t-\alpha)h(\alpha)\,d\alpha = \mu_X \int_{-\infty}^{\infty} h(\alpha)\,d\alpha$$

So $\mu_Y(t)$ does not depend on t.

For R_{YY}:

$$R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} \underbrace{R_{XX}(t_1-\alpha, t_2-\beta)}_{R_X(t_2-\beta-t_1+\alpha)} h(\alpha)h(\beta)\, d\alpha\, d\beta$$

So $$R_{YY}(t_1,t_2) = \iint_{\mathbb{R}^2} R_X(\tau-\beta+\alpha)h(\alpha)h(\beta)\, d\alpha\, d\beta,$$

where $\tau = t_2 - t_1$

$$= R_Y(\tau)$$

Important result: If the input $X(t)$ to a stable $\left(\int_{-\infty}^{\infty} h(\alpha)\, d\alpha < \infty\right)$ LTI system is WSS, then the output $Y(t)$ is WSS

Can write R_Y more compactly:

$$R_Y(\tau) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} R_X(\tau+\alpha-\beta)\,h(\beta)\,d\beta\right]}_{R_X(\tau+\alpha) * h(\alpha)} h(\alpha)\,d\alpha$$

Substituting $\lambda = -\alpha$,

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(-\lambda)\, \underbrace{(R_X * h)(\tau-\lambda)}\, d\alpha$$

So $\boxed{R_Y(\tau) = (\tilde{h} * h * R_X)(\tau)}$,

$$\tilde{h}(t) = h(-t)$$