Probabilistic characterization of a rp

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We can define an nth ~~ordr~~ order cdf for a rp $X(t)$ as

$$F_{X(t_1)\cdots X(t_n)}(x_1,\dots,x_n;t_1,\dots,t_n) =$$

$$P(X(t_1)\le x_1,\dots,X(t_n)\le x_n)$$

for any $n\in\mathbb{N}$, ~~and~~ any $(t_1,\dots,t_n)^T\in\mathbb{R}^n$ and any $(x_1,\dots,x_n)^T\in\mathbb{R}^n$

In practice, it is ~~norm~~ often not possible to model the process this way. Instead, in many cases we use first- and second-order moment functions. ~~instead.~~

Defn. The <u>mean function</u> of a rp $X(t)$ is $\mu_X(t) = E[X(t)]\quad \forall t\in T$

Note $E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$

for each fixed $t \in T$,

where $f_{X(t)}$ is the pdf of the rv $X(t)$.

Defn. The <u>autocorrelation function</u> of a rp $X(t)$ is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)], \ \forall t_1, t_2 \in T$$

$$= \iint_{\mathbb{R}^2} x_1 x_2 \underbrace{f_{X(t_1)X(t_2)}}_{\text{joint pdf of } X(t_1),\, X(t_2)}(x_1, x_2)\,dx_1\,dx_2$$

Defn. The <u>autocovariance function</u> of a rp $X(t)$ is

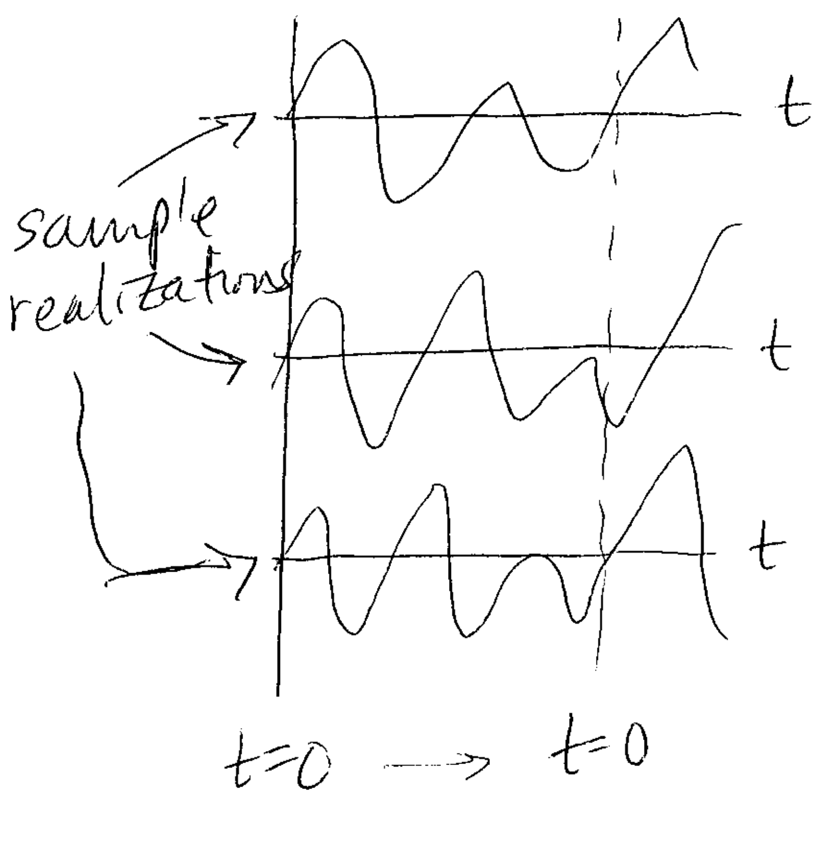
$$C_{XX}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$$

Note: If $X(t)$ is a complex-valued rp, we use

$$R_{XX}(t_1,t_2) = E[X(t_1)\,X^*(t_2)]$$

Stationarity

Do the probability distribution and mean/autocorrelation fns of a rp depend on where the time origin is?



If the point $t=0$ changes, do the pdfs change? What about R_{XX} and C_{XX}, and μ_X?

Two types of stationarity will be discussed.

Defn. A rp $X(t)$ is <u>strict-sense stationary</u> (SSS) if

$$F_{X(t_1)\cdots X(t_n)}(x_1,\dots,x_n) = F_{X(t_1+\tau)\cdots X(t_n+\tau)}(x_1,\dots,x_n)$$

$\forall n \in \mathbb{N}$; ~~all~~ $\forall (t_1,\dots,t_n)^T \in T^n$; $\forall x_1,\dots,x_n$; for any $\tau \in T$

Note: To ~~show~~ determine whether a rp is SSS, compute the rhs. If it does not depend on τ, then $X(t)$ is SSS, otherwise $X(t)$ is not SSS.

Note that:

- If $X(t)$ is SSS, then

$\cancel{f_{X(t)}\ f_{X(t)}(x) = f_X(t+\tau}$

$$f_{X(t)}(x) = f_{X(t+\tau)}(x) = f_X(x)$$

for any t, τ, and $x \in \mathbb{R}$
and some pdf f_X

- If $X(t)$ is SSS, then

$$f_{X(t_1)X(t_2)}(x_1, x_2) =$$

$$f_{X(t_1+\alpha)X(t_2+\alpha)}(x_1, x_2) =$$

$$f_{X_1X_2}(x_1, x_2, \tau), \ \forall t_1, t_2, x_1, x_2, \alpha$$

where

$$\tau = t_2 + \alpha - (t_1 + \alpha) = t_2 - t_1$$

and $f_{X_1X_2}$ is a second order pdf

Defn. A rp $X(t)$ is wide-sence stationary (WSS) if it satisfies: (6)

(1) $E[X(t)] = \mu_X \quad \forall t \in T$ for some $\mu_X \in \mathbb{R}$

(2) $R_{XX}(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$ for some function $R_X : \mathbb{R} \to \mathbb{R}$, where $\tau = t_2 - t_1$, for all $t_1, t_2 \in T$

Some properties:

- If $X(t)$ is WSS, then
  - $E[X^2(t)] = R_{XX}(t,t) = R_X(0)$, and this means that $R_X(0) \geq 0$
  - $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$
    $= R_X(\tau) - \mu_X^2 = C_X(\tau), \tau = t_2 - t_1$

- If $X(t)$ is SSS, then it is WSS, but the converse is not necessarily true. Proof omitted

Defn. A rp $X(t)$ is a <u>Gaussian rp</u> if $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian rvs, $\forall n \in \mathbb{N}$ and any $(t_1, \ldots, t_n)^T \in T^n$

The nth-order char. fn of a Gaussian rp is

$$\Phi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) =$$

$$\exp\left[i \sum_{k=1}^{n} \mu_X(t_k)\omega_k - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_{XX}(t_j, t_k)\omega_j\omega_k\right],$$

for $n \in \mathbb{N}$, $(t_1, \ldots, t_n)^T \in T^n$, $(\omega_1, \ldots, \omega_n)^T \in \mathbb{R}^n$