Notes presented on chalkboard 11/15/2022

Theorem. (Strong Law of Large Numbers) Let X_n be a sequence of iid rvs with mean μ and variance σ^2. Then if the mean and variance are finite,

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.e.} \mu \quad \text{as } n \to \infty$$

Proof. Omi~~tt~~ted.

So for the example

$$X_k = s + W_k$$

if the W_k's are independent and have the same variance σ^2, then the sample mean Y_n converges to s with probability 1.

But the laws of large numbers tell us something more fundamental about our axiomatic-based prob. measure. Consider a rv X and a set $A \subset \mathcal{B}(\mathbb{R})$. Let $X_1, X_2, \ldots$ be an iid sequence of rvs with the same distribution as X. Now let

$$Y_k = \begin{cases} 1 & \text{if } X_k \in A \\ 0 & \text{if } X_k \notin A \end{cases}$$

Then $E[Y_k] = P(X_k \in A) = P(X \in A)$

The Y_k's are iid with mean $P(X \in A)$, and $\frac{1}{n}\sum_{k=1}^{n} Y_k$ is the relative frequency of the event $\{X \in A\}$. The SLLN tells us that

$$\frac{1}{n}\sum_{k=1}^{n} Y_k \longrightarrow P(X \in A) \text{ wp1, as } n \to \infty$$

So with prob. 1, the relative frequency approach to prob. converges to the axiom-base $P(X \in A)$.

The Central Limit Theorem

Let X_n be a sequence of iid rvs with finite mean μ and finite variance σ^2. Then if

$$Z_n = \frac{\sum_{j=1}^{n} X_j - n\mu}{\sigma\sqrt{n}}$$

then $Z_n \xrightarrow{d} Z$ as $n \to \infty$

where Z is $N(0,1)$

i.e., $F_{Z_n}(z) \longrightarrow \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

Proof. Will use without proof the following:

Lemma. Let Z_n be a sequence of rvs with cdfs F_{Z_n} and mgfs $\Phi_{Z_n}(s)$, and let Z be a rv with cdf F_Z and mgf Φ_Z. Then if $\Phi_{Z_n}(s) \to \Phi_Z(s)$

$\forall s \in \mathbb{C}$, then $F_{Z_n}(z) \to F_Z(z)$ $\forall z \in \mathbb{R}$.

So it is sufficient to show that $\Phi_{Z_n}(s) \longrightarrow e^{s^2/2}$

Consider $\Phi_X\left(\frac{s}{\sqrt{n}}\right) = E\left[e^{sX_k/\sqrt{n}}\right]$ $\forall k$

Now considering the case $\mu = 0, \sigma^2 = 1$, we have

$$\Phi_{Z_n}(s) = E\left[e^{s\sum_{k=1}^{n} X_k/\sqrt{n}}\right]$$

$$= E\left[\prod_{k=1}^{n} e^{sX_k/\sqrt{n}}\right] = \prod_{k=1}^{n} E\left[e^{sX_k/\sqrt{n}}\right]$$

$$= \prod_{k=1}^{n} \Phi_X\left(\frac{s}{\sqrt{n}}\right) = \left(\Phi_X\left(\frac{s}{\sqrt{n}}\right)\right)^n$$

Let $L(s) = \log \Phi_X(s)$. Then to show $\Phi_{Z_n}(s) \longrightarrow e^{s^2/2}$, it is sufficient to show

$$nL\left(\frac{s}{\sqrt{n}}\right) = \log\left[\left(\Phi_X\left(\frac{s}{\sqrt{n}}\right)\right)^n\right] \to \frac{s^2}{2}$$

Will need:

$$L(0) = 0$$

$$L'(0) = \mu = 0$$

$$L''(0) = E[X^2] = 1$$

Now

$$\lim_{n\to\infty} nL\left(\frac{s}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{L\left(\frac{s}{\sqrt{n}}\right)}{n^{-1}}$$

$$= \lim_{n\to\infty} \frac{-L'\left(\frac{s}{\sqrt{n}}\right) n^{-3/2} s}{-2n^{-2}} \quad \text{(l'hopital)}$$

$$= \lim_{n\to\infty} \frac{L'\left(\frac{s}{\sqrt{n}}\right) s}{2n^{-1/2}}$$

$$= \lim_{n\to\infty} \frac{-L''\left(\frac{s}{\sqrt{n}}\right) n^{-3/2} s^2}{-2n^{-3/2}} \quad \text{(l'hopital again)}$$

$$= \lim_{n\to\infty} \frac{L''\left(\frac{s}{\sqrt{n}}\right) s^2}{2} = \frac{s^2}{2}$$

For general μ, σ^2, use result for $\mu = 0, \sigma^2 = 1$ with $Y_k = \frac{X_k - \mu}{\sigma}$