11/1/2022

Regrade requests must be submitted by Thurs 11/3 at 11:59pm

Note the difference:

- $E[Y|X=x]$ depends on $x \in \mathbb{R}$
- $E[Y|X]$ is a rv created by letting
$$h(x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy = E[Y|X=x],$$
and then creating the rv $h(X)$
- Recall: do not use notation $E[Y|x]$

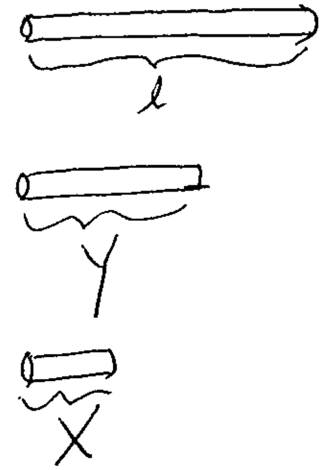
We can now write

$$E[Y] = \underline{\underline{E\big[E[Y|X]\big]}} = \int h(x) f_X(x)\,dx$$

Could also write

$$E[Y] = E_X\left[E_{Y|X}[Y|X]\right]$$

Example. Have a stick of length $l \in \mathbb{R}$. Break the stick at a uniformly chosen point Y, then again at a uniformly chosen point X.



Find $E[X]$.

$$E[X] = \iint_{\mathbb{R}^2} x f_{XY}(x,y)\,dy\,dx$$

$$= \int_{\mathbb{R}} x f_X(x)\,dx$$

We know $f_Y(y) = \frac{1}{l}, \quad 0 \le y \le l$

$$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 \le x \le y$$

Using iterated expectation,

$$E[X] = E[E[X|Y]]$$

first find $h(y)$.

$$h(y) = E[X \mid Y = y] = \frac{y}{2}$$

so $E[X|Y] = h(Y) = \frac{Y}{2}$, so

$$E[X] = E[h(Y)] = E\left[\frac{Y}{2}\right] = \frac{1}{2}\frac{l}{2}$$
$$= \frac{l}{4}$$

Note that normally expectation gives a real number, but $E[X|Y]$ is a rv.

Random Vectors

Often in practice, we have more ~~that~~ than 2, but still a finite number of rvs. We will consider n rvs $X_1, \ldots, X_n$ for finite $n \geq 1$. We can view this as a random vector

$$\underline{X} = [X_1, \ldots, X_n]^T$$

A ~~RV has~~ RV (random vector) has a joint cdf

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, \dots, X_n \le x_n),$$

where $\underline{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$

and a joint pdf that satisfies

$$P(\underline{X} \in D) = \int_D f_{\underline{X}}(\underline{x})\, d\underline{x}$$

for $D \in \mathcal{B}(\mathbb{R}^n)$

One important class of RVs is those that satisfy a Markov Property, for example,

$$f_{X_n|X_{n-1}\cdots X_1}(x_n | x_1, \dots, x_{n-1}) = f_{X_n|X_{n-1}}(x_n | x_{n-1})$$

Often we use first and second order moments to characterize a RV.

Defn. The <u>mean vector</u> $\underline{\mu}_{\underline{X}}$ of RV $\underline{X}$ is

$$\underline{\mu}_{\underline{X}} = E[\underline{X}] = [E[X_1], \cdots, E[X_n]]^T$$

Recall that the correlation of two rvs X_j and X_k is

$$E[X_j X_k], \quad j \neq k$$

Defn. The <u>correlation matrix</u> of RV $\underline{X}$ is

$$R_{\underline{X}} = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & & \\ R_{n1} & \cdots & R_{nn} \end{bmatrix}$$

where $R_{ii} = E[X_i^2]$ and

$R_{jk} = E[X_j X_k]$, $j \neq k$.

Also the <u>covariance matrix</u> $C_{\underline{X}}$ has elements $C_{jk} = E[(X_j - \bar{X}_j)(X_k - \bar{X}_k)]$

Its diagonal elements are variances.

Its off-diagonal elements
are covariances

Both $R_{\underline{X}}$ and $C_{\underline{X}}$ are <u>non-negative definite</u> (NND) matrices

Note that an $n \times n$ matrix B with elements b_{ij} is NND if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j b_{ij} \geq 0 \quad \forall [x_1, \ldots, x_n]^T \in \mathbb{R}^n$$

Proof omitted