Considering $Y=g(X)$ when Oct 4, 2022
X is cont. and Y is discrete
(case 2)

We can let $\mathcal{R}_Y = g(\mathbb{R})$
find the pmf of Y.

We know that

$$p_Y(y) = P(g(X)=y) = P(X \in D_y)$$

where $D_y = \{x \in \mathbb{R} : g(x) = y\}, \forall y \in \mathcal{R}_Y$

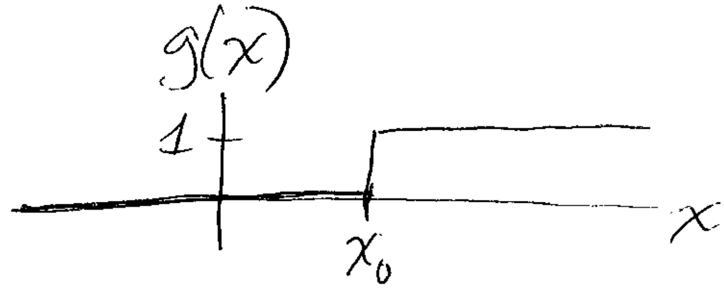
So we need to find D_y such that

$$\{g(X)=y\} = \{X \in D_y\} \text{ for each } y \in \mathcal{R}_Y$$

Then $p_Y(y) = \int_{D_y} f_X(x)\,dx$

Example. Let $g(x) = I_{[x_0,\infty)}(x)$

indicator function



Can see that $R_Y = \{0, 1\}$

For $\cancel{x \le x_0}$ $y = 0$:

$$D_0 = \{x \in \mathbb{R} : g(x) = 0\} = (-\infty, x_0)$$

For $y = 1$:

$$D_1 = \{x \in \mathbb{R} : g(x) = 1\} = [x_0, \infty)$$

So $$p_Y(0) = \int_{-\infty}^{x_0} f_X(x)\,dx$$

$$p_Y(1) = \int_{x_0}^{\infty} f_X(x)\,dx$$

$$p_Y(y) = \begin{cases} \int_{-\infty}^{x_0} f_X(x)\,dx & \text{if } y = 0 \\ \int_{x_0}^{\infty} f_X(x)\,dx & \text{if } y = 1 \end{cases}$$

Case 3 X, Y are both discrete

Can let $R_Y = g(\mathbb{R})$ or

$R_Y = g(R_X)$

If you know R_X, use $g(R_X)$, if not, use $g(\mathbb{R})$

For this case

$$p_Y(y) = P(g(X) = y)$$
$$= P(\{x \in R_X : g(x) = y\})$$

(or $x \in \mathbb{R}$ if R_X is not known)

So

$$p_Y(y) = \sum_{\substack{x \in R_X : \\ g(x) = y}} p_X(x), \ \forall y \in R_Y$$

(can write
$D_y = \{x \in R_X : g(x) = y\}$)

Example. Let X be the value rolled on a die, and

$$Y = \begin{cases} 1 & \text{if } X \text{ is odd} \\ 0 & \text{if } X \text{ is even} \end{cases}$$

We have

$$R_X = \{1,2,3,4,5,6\}$$
$$R_Y = \{0,1\}$$

Note that $g(x) = x \% 2$

We then have

$$p_Y(y) = \sum_{\substack{x \in R_X: \\ x \% 2 = y}} p_X(x), \quad y \in \{0,1\},$$

or

$$p_Y(0) = p_X(2) + p_X(4) + p_X(6)$$
$$p_Y(1) = p_X(1) + p_X(3) + p_X(5)$$

Expectation

Expectation is often used in probability for:

- characterizing a rv (or a collection of rvs) when there is insufficient data to find histograms
- finding the parameters

for many types of rvs (e.g., μ, σ for a Gaussian)

Defn. The <u>expected value</u> of a rv X is defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \in \mathbb{R}$$

Also called the <u>mean</u> of X

Other notation: $\mu, \mu_X, EX, \overline{X}$

It can be shown (using the sifting property) that

$$E[X] = \sum_{x \in R_X} x\, p_X(x) \quad \text{if } X \text{ is discrete}$$

Example. Let X be exponential with parameter λ, so

$$f_X(x) = \lambda e^{-\lambda x} u(x), \quad \lambda > 0$$

So $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \lambda \int_0^{\infty} x e^{-\lambda x}\,dx$$

which gives

$$E[X] = \frac{1}{\lambda}$$

letting this value be denoted μ gives

$$f_X(x) = \frac{1}{\mu} e^{-x/\mu} u(x)$$