We are considering 10/27/2022

$$F_Y(y \mid x_1 < X \leq x_2) = \frac{P(Y \leq y,\ x_1 < X \leq x_2)}{P(x_1 < X \leq x_2)}$$

$$= \frac{F_{XY}(x_2, y) - F_{XY}(x_1, y)}{F_X(x_2) - F_X(x_1)}$$

So

$$f_Y(y \mid x_1 < X \leq x_2) =$$

$$\frac{\frac{\partial}{\partial y}\left[\int_{-\infty}^{x_2}\int_{-\infty}^{y} f_{XY}(x, y')\,dx\,dy' - \int_{-\infty}^{x_1}\int_{-\infty}^{y} f_{XY}(x, y')\,dx\,dy'\right]}{F_X(x_2) - F_X(x_1)}$$

$$= \frac{\int_{x_1}^{x_2} f_{XY}(x, y)\,dx}{F_X(x_2) - F_X(x_1)}$$ (using Calculus to simplify)

In practice want $f_Y(y \mid X = x)$.

Define $f_Y(y \mid X = x) = \lim_{\Delta x \to 0} f_Y(y \mid x < X \leq x + \Delta x)$

Using the result for $f_Y(y|x_1 < X \le x_2)$ with $x_1 = x$ and $x_2 = x + \Delta x$, and a similar procedure to that used to find $P(B|X=x)$ earlier,

$$f_Y(y|X=x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Notation: Usually use either $f_Y(y|X=x) = f_{Y|X}(\cancel{x|y}\ y|x)$ or simply $f(y|x)$

Similarly, $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

This ~~is~~ leads to

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$$

$$(\text{if } f_X(x) \neq 0,\ f_Y(y) \neq 0)$$

This is another form of Bayes' Theorem

Also, there is a version of the TPL,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx =$$

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_X(x)\,dx$$

Summary of Bayes' Thm for Two rvs

1. X, Y both discrete

Use $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

with $A = \{X = x\}$, $B = \{Y = y\}$, $\forall x \in \mathcal{R}_X, y \in \mathcal{R}_Y$ with $P(X=x) > 0$, $P(Y=y) > 0$

Notation: $p_{X|Y}(x|y) = P(X=x \mid Y=y)$

conditional pmf of X given Y.

We can write

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\, p_X(x)}{p_Y(y)}$$

(2) X discrete, Y continuous:

Use $P(A|Y=y) = \frac{f_Y(y|A)\,P(A)}{f_Y(y)}$

with $A = \{X = x\}$, $x \in R_X$

We write

$$p_{X|Y}(x|y) = \frac{f_Y(y|x)\,p_X(x)}{f_Y(y)}$$

(3) X, Y both continuous

Use $f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)\,f_Y(y)}{f_X(x)}$

Conditional Expectation

If X and Y are rvs defined on $(S, \mathcal{F}, P)$, then

$$E[g(X,Y)|M] \equiv \iint_{\mathbb{R}^2} g(x,y)\,f_{XY}(x,y|M)\,dx\,dy$$

Consider the case $M = \{Y = y\}$

This gives

$$E[g(X,Y)|Y=y] = \iint_{\mathbb{R}^2} g(x,y')f_{XY}(x,y'|Y=y)\,dx\,dy'$$

Can show that this is

$$E[g(X,Y)|Y=y] = E[g(X,y)|Y=y] = \int_{-\infty}^{\infty} g(x,y)f_{X|Y}(x|y)\,dx$$

Note: If X and Y are independent, then

$$f_{XY}(x,y) = f_X(x)f_Y(y) \quad \text{so}$$

$$f_{X|Y}(x|y) = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

In this case

$$E[g(X,Y)|Y=y] = E[g(X,y)]$$

If X,Y are not ind.

$$E[g(X,Y)|Y=y] = E[g(X,y)|Y=y]$$

$$\neq E[g(X,y)], \text{ in general}$$

Note $E[g(X,y)|Y=y] = \int_{-\infty}^{\infty} g(x,y)f_{X|Y}(x|y)\,dx$

An important special case:
$M = \{Y = y\}$ and $g(x,y) = g(x)$ for some $g: \mathbb{R} \to \mathbb{R}$, so $g(x,y)$ does not depend on y

In this case,

$$\boxed{E[g(X) \mid Y = y]} = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx$$

Use this notation

$E[g(X) \mid y]$ is not defined

$E[g(X) \mid Y]$ means something different

Note: $F_{X|Y}(x|y) = \underline{P(X \le x \mid Y = y)}$

$$= \int_{-\infty}^{x} f_X(x \mid Y = y)$$

$$\longrightarrow 1 \text{ as } x \to \infty$$

Iterated Expectation

Sometimes can use iterated expectation to find $E[Y]$ in an easier way than the defn. of $E[Y]$.

Start by writing

$$E[Y] = \iint_{\mathbb{R}^2} y \, f_{XY}(x, y) \, dx \, dy$$

$$= \iint_{\mathbb{R}^2} y \, f_{Y|X}(y|x) \, f_X(x) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy \right]}_{E[Y|X=x]} f_X(x) \, dx$$