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The Cauchy-Schwarz Inequality (CSI)

For two rvs X, Y, the CSI states that

$$|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$$

with equality iff $Y = a_0 X$, for some $a_0 \in \mathbb{R}$, with prob. 1.

Proof. Consider $0 \le E[(aX-Y)^2] =$

$$E[X^2]a^2 - 2aE[XY] + E[Y^2]$$

for ~~each~~ arbitrary $a \in \mathbb{R}$, $a \neq 0$

Can view this as a quadratic function

Case ① $E[(aX-Y)^2] > 0$

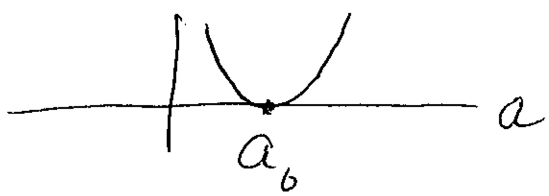
In this case the roots are complex

The quadratic formula gives

$$4\left(E[XY]\right)^2 - 4E[X^2]E[Y^2] < 0$$

or $$|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$$

Case 2: $E[(aX-Y)^2] = 0$



There is a real number a_0 satisfying $E[\underbrace{(a_0X-Y)}_{\text{rv with mean-squared value } 0}{}^2] = 0$

It can be shown that if a rv X has $E[X^2]=0$, then $X=0$ except possibly on a set of prob. 0. In other words, there is a set A in $\mathcal{F}$ with prob. 1 such that

$$X(\omega) = 0 \quad \forall \omega \in A.$$

So $a_0 X - Y = 0$ w.p.1,

or $Y = a_0 X$ with p1.

Joint Moments

Defn. The *joint moments* of X and Y are

$$E[X^j Y^k] \quad \text{for } j = 0, 1, \ldots; \; k = 0, 1, \ldots$$

and the *joint central moments* are

$$E[(X-\bar{X})^j (Y-\bar{Y})^k], \quad j = 0, 1, \ldots; \; k = 0, 1, \ldots$$

If $j=1$, $k=0$, then:

$$E[X], \quad E[(X-\bar{X})] = 0$$

If $j=2$, $k=0$, then:

$$E[X^2], \quad E[(X-\bar{X})^2] \sim \text{mean-squared, variance}$$

If $j=k=1$, then:

$$E[XY], \quad E[(X-\bar{X})(Y-\bar{Y})] \quad \text{correlation, covariance}$$

Defn. The joint moment-generating function (mgf) of X and Y is

$$\phi_{XY}(s_1, s_2) = E\left[e^{s_1 X + s_2 Y}\right], \quad \forall s_1, s_2 \in \mathbb{R}$$

$$= \iint_{\mathbb{R}^2} e^{s_1 x + s_2 y} f_{XY}(x,y)\, dx\, dy$$

The joint characteristic function of X and Y is

$$\Phi_{XY}(\omega_1, \omega_2) = \phi_{XY}(s_1, s_2)\Big|_{s_1 = i\omega_1,\ s_2 = i\omega_2}$$

$$\forall \omega_1, \omega_2 \in \mathbb{R}$$

So $$\Phi_{XY}(\omega_1, \omega_2) = \iint_{\mathbb{R}^2} \exp\left[i\omega_1 x + i\omega_2 y\right] f_{XY}(x,y)\, dx\, dy$$

Some comments:

- When $\omega_1 = \omega$, $\omega_2 = 0$ for some $\omega \in \mathbb{R}$, we get $\Phi_{XY}(\omega, 0) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\, dx$

This is the char. fn. of X:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x)\,dx,$$

$$\omega \in \mathbb{R}$$

- Φ_X, Φ_Y are 1D "Fourier-transform-like", except for the missing negative sign in the exp.
- If $Z = aX + bY$, then

$$\Phi_Z(\omega) = \Phi_{XY}(a\omega, b\omega)$$

$$\forall \omega, a, b \in \mathbb{R}$$

- If X, Y are independent then $\Phi_{XY}(\omega_1, \omega_2) = \Phi_X(\omega_1)\Phi_Y(\omega_2)$

$$\forall \omega_1, \omega_2 \in \mathbb{R}$$

The converse is also true

- If X, Y are iid and $Z = X + Y$, then

$$\Phi_Z(\omega) = \Phi_X(\omega)\,\Phi_Y(\omega)$$

$$= \left(\Phi(\omega)\right)^2 \quad \text{where}$$

Φ is the char. fn. of X, Y.

Proofs are left to the students

The <u>Moment Generating Theorem</u>

This theorem can be used to compute moments using the result

$$E[X^j Y^k] = \frac{\partial^j}{\partial s_1^j} \cdot \frac{\partial^k}{\partial s_2^k} \phi_{XY}(s_1, s_2)\Big|_{s_1 = s_2 = 0}$$

Proof for the case $j = n, k = 0$ for some $n \geq 1$. Show that

$$E[X^n] = \frac{d^n}{ds^n} \phi_X(s)\Big|_{s=0}$$

We have

$$\frac{d^n}{ds^n}\phi_X(s) = \frac{d^n}{ds^n} E\left[e^{sX}\right]$$

$$= E\left[\frac{d^n}{ds^n} e^{sX}\right] = E\left[X^n e^{sX}\right]$$

So $$\left.\frac{d^n}{ds^n}\phi_X(s)\right|_{s=0} = E[X^n]$$

In terms of the char. fn.:

$$E[X^n] = \frac{1}{i^n}\frac{d^n}{d\omega^n}\Phi_X(\omega)\Big|_{\omega=0}$$

Conditional Distributions for Two rvs

Sometimes we can model a conditional and a marginal distr. more easily than a joint distr. Also, sometimes we need a conditional distr. for our task, instead of a joint or marginal.

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Defn. For rvs X, Y on $(S, \mathcal{F}, P)$, the cdf of X and Y given an event $M \in \mathcal{F}$ is

$$F_{XY}(x,y|M) = \frac{P(\{X \le x\} \cap \{Y \le y\} \cap M)}{P(M)}, \text{ if } P(M) > 0$$

We are most interested in the conditional density function of Y given an observed value of X.

First consider

$$F_Y(y | x_1 < X \le x_2) =$$

$$\lim_{x \to \infty} F_{XY}(x,y|M) \text{ where}$$

$$M = \{x_1 < X \le x_2\} \text{ for}$$

$$x_1 < x_2, \ x_1, x_2 \in \mathbb{R}$$