Two Functions of Two rvs

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Given two rvs X, Y, consider

$$Z = g(X,Y)$$
$$W = h(X,Y)$$

where $g: \mathbb{R}^2 \to \mathbb{R}$, $h: \mathbb{R}^2 \to \mathbb{R}$

For example,

$$\begin{bmatrix} Z \\ W \end{bmatrix} = A_{2\times 2} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

This can be used to represent rotation, scaling, projection in 2D, for example. In this case,

$$g(x,y) = a_{11}x + a_{12}y$$
$$h(x,y) = a_{21}x + a_{22}y$$

Find f_{ZW}. We could find

$$F_{ZW}(z,w) = P((X,Y) \in D_{zw})$$

for some $D_{zw} \in \mathcal{B}(\mathbb{R}^2)$, but finding D_{zw} in general is very difficult.

Instead, use a change-of-variables approach:

Assume that:

- the system

$$z = g(x,y)$$
$$w = h(x,y)$$

can be solved simultaneously for

$$x = g^{-1}(z,w)$$
$$y = h^{-1}(z,w)$$

For the linear transf. example, this means A_{22} is invertible, in which case

$$g^{-1}(z,w) = b_{11} z + b_{12} w$$
$$h^{-1}(z,w) = b_{21} z + b_{22} w$$

elements of A_{22}^{-1}

- $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ all exist

Under these two assumptions, (3)

$$f_{ZW}(z,w) = \frac{f_{XY}(g^{-1}(z,w), h^{-1}(z,w))}{\left| \frac{\partial(z,w)}{\partial(x,y)} \right|}$$

Jacobian of the transformation

$$\left|\frac{\partial(z,w)}{\partial(x,y)}\right| \equiv \left| \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \right| = \left| \frac{\partial z}{\partial x} \cdot \frac{\partial w}{\partial y} - \frac{\partial z}{\partial y} \frac{\partial w}{\partial x} \right|$$

absolute value of the determinant

Proof in Papoulis

This results assumes the Jacobian is not 0

Example. X and Y are iid Gaussian with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = \sigma$, $r = 0$ (parameter of joint Gaussian pdf)

Let $R = \sqrt{X^2+Y^2}$, $\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$

so $g(x,y) = \sqrt{x^2+y^2}$, $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

The formula for $f_{R\Theta}$ gives (4)

$$f_{R\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) u(r), \text{ for}$$

$$-\pi \le \theta \le \pi$$

We can find $f_R(r)$ by integrating:

$$f_R(r) = \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) d\theta, \quad r \ge 0$$

$$= \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) u(r)$$

This is called the Rayleigh r.v.

Note that if $Z = g(X,Y)$, can find $f_Z(z)$ by defining an 'auxiliary' variable W, find f_{ZW}, then integrating f_{ZW} over w. Sometimes a good choice is $W = X$ or $W = Y$.

Joint Expectation

Given rvs X, Y, and $Z = g(X,Y)$ for some $g: \mathbb{R}^2 \rightarrow \mathbb{R}$, we know that

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz$$

from the defn. of exp. of a rv.

It can be shown that

$$E[Z] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

Proof in Papoulis

If X, Y are discrete, can use

$$E[Z] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x,y)\, p_{XY}(x,y)$$

joint pmf

Some important joint expectations:

- The <u>correlation</u> between X and Y is

$$\mathrm{Corr}(X,Y) \equiv E[XY]$$

- The <u>covariance</u> of X and Y is

$$\mathrm{Cov}(X,Y) \equiv E[(X-\bar{X})(Y-\bar{Y})]$$ 6

- The correlation coefficient of X and Y is

$$r_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Note:

— Sometimes called r or ρ

— $|r_{XY}| \le 1$

Proof on HW

— If X and Y are ind., then $r_{XY} = 0$.

$$\mathrm{Cov}(X,Y) = E[(X-\bar{X})(Y-\bar{Y})] =$$

$$\iint (x-\bar{X})(y-\bar{Y}) f_X(x) f_Y(y)\,dx\,dy$$

$$= \underbrace{\int_{-\infty}^{\infty}(x-\bar{X})f_X(x)\,dx}_{\bar{X}-\bar{X}} \underbrace{\int_{-\infty}^{\infty}(y-\bar{Y})f_Y(y)\,dy}_{\bar{Y}-\bar{Y}}$$

So cov = 0, and $r_{XY} = 0$

If $r_{XY} = 0$, then 7
X, Y are not necessarily ind.

- If $r_{XY} = 0$, then X, Y are said to be <u>uncorrelated</u>

Equivalent definitions for uncorrelated rvs are:

- X and Y are uncorrelated iff $Cov(X,Y) = 0$, or
- X, Y are uncorrelated iff $E[XY] = \bar{X}\bar{Y}$

Final definition: X and Y are <u>orthogonal</u> if $E[XY] = 0$