We have defined the joint cdf and joint pdf of two rvs X and Y. If X and Y are both discrete, we use: 10/18/2022

Defn. the <u>joint probability mass function</u> of two discrete rvs X and Y is

$$p_{XY}(x,y) = P(\{X=x\} \cap \{Y=y\}),$$
$$\forall x \in \mathcal{R}_X,\ y \in \mathcal{R}_Y$$

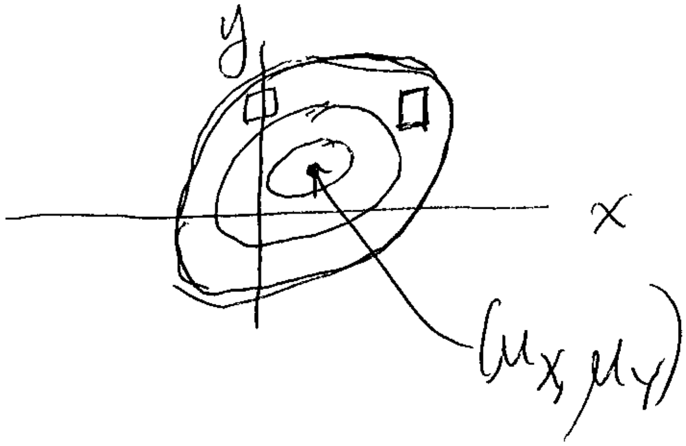
<u>Jointly Gaussian Random Variables</u>

Two rvs X and Y are jointly Gaussian if

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2r(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right]$$

where $\mu_X, \mu_Y, \sigma_X, \sigma_Y, r \in \mathbb{R}$; $\sigma_X > 0$, $\sigma_Y > 0$, $-1 < r < 1$

Note that the contours of this function are ellipses



The major and minor axes (lengths and orientations) depend on σ_X, σ_Y, r

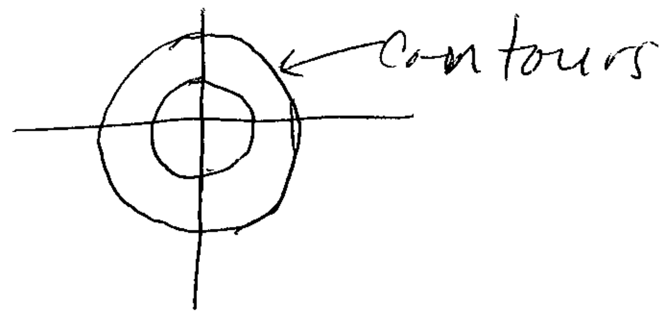
It can be shown that if X, Y are jointly Gaussian, then X and Y are each marginally Gaussian

— The converse is not necessarily true

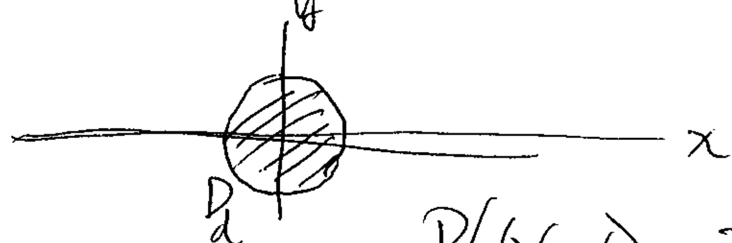
Example. Consider jointly Gaussian rvs X and Y with

$\mu_X = \mu_Y = r = 0, \quad \sigma_X = \sigma_Y = \sigma \in \mathbb{R}$

Then $f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$

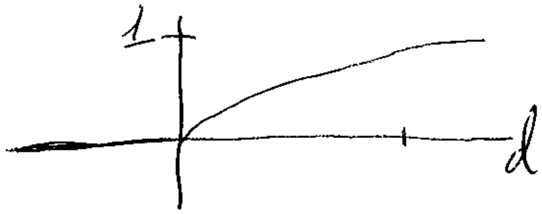


Find the probability that (X,Y) lies within a distance d of the origin



$P((X,Y) \in D_d)$?

$$P((X,Y)\in D_d) = \int_{D_d} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) dx\,dy$$



$$= 1 - e^{-d^2/2\sigma^2}$$

Statistical Independence of Two rvs

Defn. Two rvs X and Y are statistically independent if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent $\forall A, B \in \mathcal{B}(\mathbb{R})$, ie.

$$P(\{X \in A\} \cap \{Y \in B\}) = P(X \in A) P(Y \in B)$$

Equivalent definition:

X and Y are independent if $F_{XY}(x,y) = F_X(x) F_Y(y)$ $\forall x, y \in \mathbb{R}$

To show equivalence:

First, assume X, Y are ind. and show $F_{XY}(x,y) = F_X(x) F_Y(y)$

Let $A = (-\infty, x]$, $B = (-\infty, y]$ for any $x, y \in \mathbb{R}$

Then $F_{XY}(x,y) = P(X \in A, Y \in B)$

$= P(X \in A) P(Y \in B)$

So $F_{XY}(x,y) = P(X \le x)P(Y \le y)$

$= F_X(x) F_Y(y)$,

so $f_{XY}(x,y) = f_X(x) f_Y(y)$

Proof of the converse is left to the student.

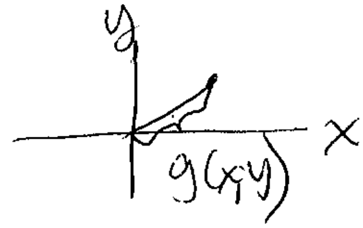
One Function of Two rvs

Given rvs X, Y and a function $g: \mathbb{R}^2 \to \mathbb{R}$, let $Z = g(X,Y)$.

What is $f_Z(z)$?

Some common examples:

- $g(x,y) = x+y$
- $g(x,y) = \frac{1}{2}(x+y)$
- $g(x,y) = \sqrt{x^2+y^2}$
- $g(x,y) = A_{2\times 2}\begin{bmatrix} x \\ y \end{bmatrix}$



General approach:

Find $F_Z(z)$ first, then differentiate to get f_Z

$$F_Z(z) = P(Z \le z) = P(g(X,Y) \le z)$$

$$= P((X,Y) \in D_z) \quad \text{where}$$

$$D_z \in \mathcal{B}(\mathbb{R}^2)$$

Need to find, for each $z \in \mathbb{R}$, $D_z = \{(x,y) \in \mathbb{R}^2 : g(x,y) \le z\}$

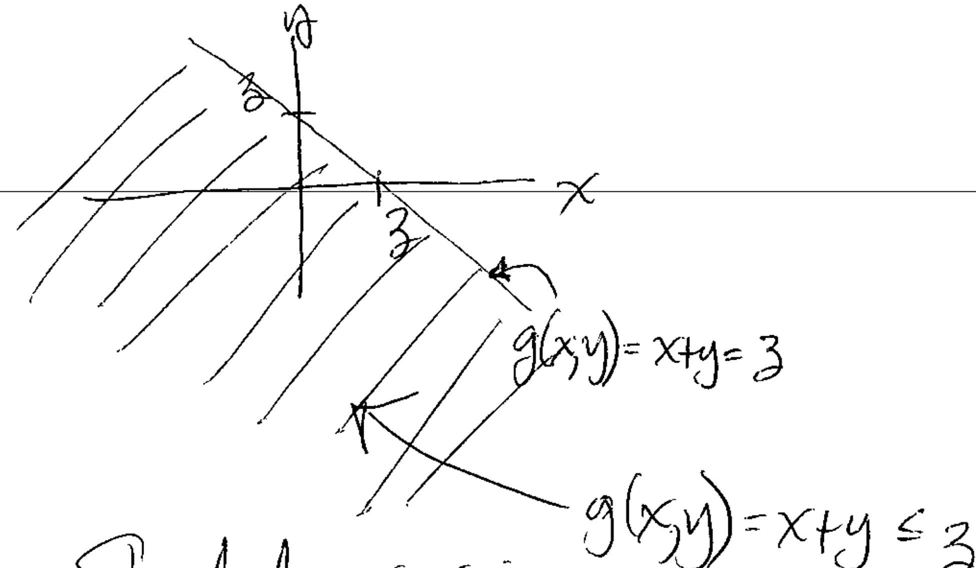
Then $F_Z(z) = P((X,Y) \in D_z)$

$$= \int_{D_z} f_{XY}(x,y)\,dx\,dy$$

Example. $Z = X + Y$

$g(x,y) = x + y$ here

$$P(Z \le z) = P(X + Y \le z)$$



Shaded region is D_z

So $$F_Z(z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

This is the result for general X,Y.

If we assume X,Y are ind, then

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} f_X(x) F_Y(z-x)\,dx$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

$$= (f_X * f_Y)(z) \quad \forall z \in \mathbb{R}$$

Example. X and Y are ind. exponential rvs with $\mu_x = \mu_y = \mu$.

Let $Z = X + Y$.

Then
$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\mu} e^{-x/\mu} u(x) \cdot \frac{1}{\mu} \cancel{e^{-y/\mu}} e^{-(z-x)/\mu} u(z-x)\, dx$$

$$= \frac{z}{\mu^2} e^{-z/\mu} u(z)$$