See the syllabus for updated HW due dates and Exam 2, Final information 10/13/2022

Expectation (cont'd)

Recall that

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

This form can be generalized to find $E[g(X)]$ for some function $g:\mathbb{R}\to\mathbb{R}$. Letting $Y=g(X)$, we could use

$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy.$$

But this requires $f_Y(y)$.

It can be shown that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Proof is in Papoulis.

Note that if

- $g(x) = x$, this result reduces to the defn. of $E[X]$
- $g(x) = (x - \bar{X})^2$, then

$$E[g(X)] = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty} (x-\bar{X})^2 f_X(x)\,dx$$

This is the <u>variance</u> of X

Linearity of Expectation:

$$E[a_1 g_1(X) + a_2 g_2(X)] = a_1 E[g_1(X)] + a_2 E[g_2(X)]$$

for every $a_1, a_2 \in \mathbb{R}$, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$

Follows directly from linearity of integration

Example. X is $N(\mu, \sigma^2)$

Then $E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$

Could use integration by parts,
or

let $r = x - \mu$. Then

$$E[X] = \underbrace{\int_{-\infty}^{\infty} \frac{r}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr}_{\text{odd fn. of } r \text{, so integral is } 0} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr}_{\text{pdf of } N(0,\sigma^2) \text{ rv, so integral is } 1}$$

$E[X] = \mu$

Can also show that

$\mathrm{Var}(X) = \sigma^2$

Useful result: $\mathrm{Var}(X) = E[X^2] - \mu^2$

Proof left to student

Example. X is Poisson with parameter λ. Then

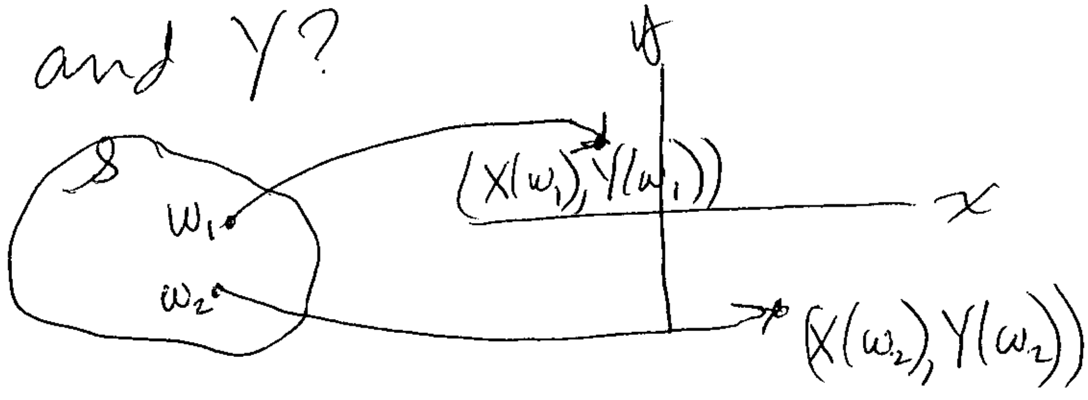
$$E[X] = \lambda$$

and $$Var(X) = \lambda$$

$$\left(p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \right)$$

Two Random Variables

Will now consider two rvs X and Y. How do we model the joint behavior of X and Y?



Instead of just asking what are $P(X \in A)$ and $P(Y \in B)$, we ask what is $P((X,Y) \in D)$

For what sets $D \subset \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ will $P((X,Y) \in D)$ be defined?

Every $D \in \mathcal{B}(\mathbb{R}^2)$, where

$\mathcal{B}(\mathbb{R}^2) = \sigma(\{\text{open rectangles in } \mathbb{R}^2\})$

If X, Y are each valid rvs, then $\{(X,Y) \in D\} \in \mathcal{F} \quad \forall D \in \mathcal{B}(\mathbb{R}^2)$

Proof omitted.

We have the joint distribution of X, Y in the ~~frmo~~ form of $P((X,Y) \in D) \; \forall D \in \mathcal{B}(\mathbb{R}^2)$, but in practice ~~at~~ often use other forms: the joint cdf, joint pdf, and joint pmf (if X, Y are discrete).

Defn. The <u>joint cumulative distribution function</u> of rvs X, Y defined on $(S, \mathcal{F}, P)$ is

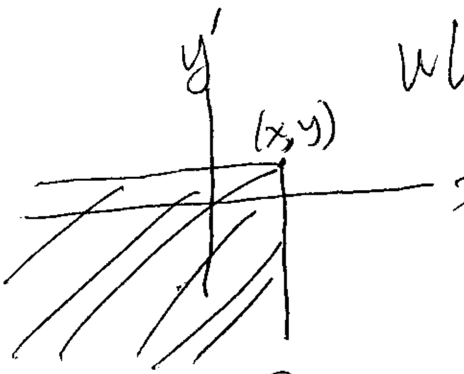
$$F_{XY}(x,y) = P(X \le x, Y \le y) \quad \forall (x,y) \in \mathbb{R}^2$$

so $F_{XY}(x,y) = P(\{X \le x\} \cap \{Y \le y\})$

Can also write as

$$F_{XY}(x,y) = P((X,Y) \in D_{xy})$$

where $D_{xy} = \{(x',y') \in \mathbb{R}^2 : x \le x', y \le y'\}$



Some properties of F_{XY}:

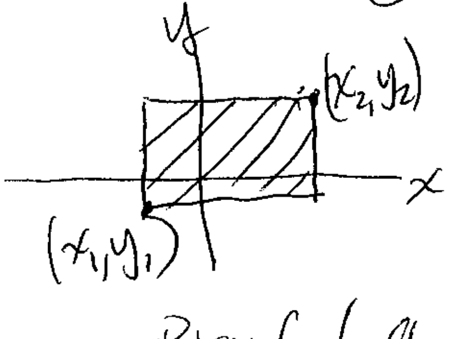
① $\lim\limits_{x \to -\infty} \lim\limits_{y \to -\infty} F_{XY}(x,y) = 0$

② $\lim\limits_{x \to \infty} F_{XY}(x,y) = F_Y(y), \ \forall y \in \mathbb{R}$

$\lim\limits_{y \to \infty} F_{XY}(x,y) = F_X(x) \quad \forall x \in \mathbb{R}$

"marginal" cdfs

③ $P(x_1 < X \le x_2, y_1 < Y \le y_2) =$



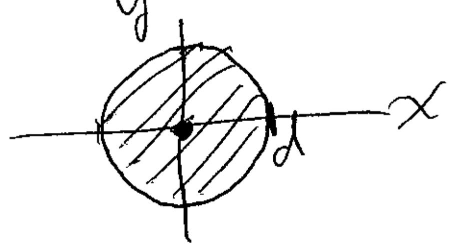
$F_{XY}(x_2,y_2) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2)$

$+ F_{XY}(x_1,y_1), \quad x_1 < x_2, \ y_1 < y_2$

Proof left to student

Note that $P((X,Y)\in D)$ for any $D \in \mathcal{B}(\mathbb{R}^2)$ could be written in terms of F_{XY} but this is often inconvenient. Consider: ⑦

$$D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le d^2\}$$ for some $d \in \mathbb{R}$, $d>0$



Writing ~~this~~ $P((X,Y)\in D)$ for this D in terms of F_{XY} is tedious.

Defn. The <u>joint probability ~~density~~ function</u> of rvs X, Y is

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{XY}(x,y)$$

for all $(x,y)\in\mathbb{R}^2$ where F_{XY} is differentiable

Could also define the joint pdf as a function f_{XY} satisfying

$$\boxed{P((X,Y)\in D) = \iint_D f_{XY}(x,y)\,dx\,dy}$$

Some Properties of the joint pdf

1. $f_{XY}(x,y) \geq 0$
2. $F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x',y')\,dx'\,dy'$ $\forall x,y \in \mathbb{R}$
3. $\iint_{\mathbb{R}^2} f_{XY}(x,y)\,dx\,dy = 1$
4. The marginal pdfs of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy, \ \forall x \in \mathbb{R}$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx, \ \forall y \in \mathbb{R}$$