Set Theory

A set is a collection of objects called elements, members, or points

Notation:

$\omega \in A$ means ω is an element of the set A

$\omega \notin A$ means ω is not in A

Two common ways to specify a set

1. Explicitly list elements

   Ex. $A = \{1,2,3,4,5,6\}$

2. Specify a rule for membership

   Ex. $A = \{\omega \in \mathbb{Z} : 1 \le \omega \le 6\}$

   the set of integers

Note that there is always a set that contains every possible element of interest. This set, along with some structure imposed upon the set, is called a <u>space</u>, denoted $\mathcal{S}$.

Some definitions we will need:

Defn. Sets A and B are <u>equal</u> iff they contain the same elements, so

$$A = B \text{ iff } w \in A \Rightarrow w \in B \text{ AND } w \in B \Rightarrow w \in A$$

If A and B are not equal, we write $A \neq B$

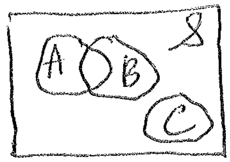
Defn. If $w \in A \Rightarrow w \in B$, then A is said to be a <u>subset</u> of B. If B contains at least one element not in A, then A is a <u>proper subset</u> of B. We will simply call A a subset of B in either case, and write $A \subset B$.

Note: $A = B$ iff $A \subset B$ AND $B \subset A$.
Prove as an exercise

Defn. The set with no elements is called the <u>empty set</u>, or <u>null set</u>, and is denoted ϕ or $\{\}$

Venn Diagrams

A Venn Diagram is a graphical representation of sets

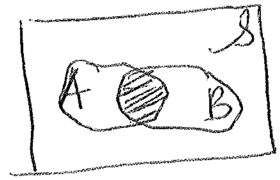
Ex. 

Note: A Venn Diagram can be useful to gain insight into a problem, but cannot be used as a proof.

Set Operations

Defn. The intersection of sets A and B is defined as

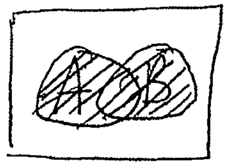
$$A \cap B = \{w \in S : w \in A \text{ and } w \in B\}$$



Shaded area represents $A \cap B$

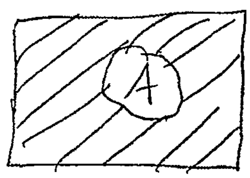
Defn. The union of sets A and B is defined as

$$A \cup B = \{w \in S : w \in A \text{ or } w \in B, \text{ or both}\}$$



Defn. The complement of a set A, denoted A^c, $\bar{A}$, or A', is defined as

$$A^c = \{w \in S : w \notin A\}$$



Defn. The set difference $A - B$ is defined as

$$A - B \equiv \{w \in S : w \in A \text{ and } w \notin B\}$$

Note: $A - B = A \cap B^c$

Defn. If sets A and B have no elements in common, i.e.,

$$A \cap B = \phi,$$

then A and B are disjoint

Algebra of Sets

1. $A \cup B = B \cup A$ Union is commutative
2. $A \cap B = B \cap A$ Intersection is commutative
3. $A \cup (B \cup C) = (A \cup B) \cup C$ $\cup$ is associative
4. $A \cap (B \cap C) = (A \cap B) \cap C$ $\cap$ is associative
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\cap$ is distributive over $\cup$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\cup$ is distributive over $\cap$
7. $(A^c)^c = A$
8. $(A \cap B)^c = A^c \cup B^c$
9. $(A \cup B)^c = A^c \cap B^c$

   (8, 9) called DeMorgan's Laws
10. $S^c = \phi$
11. $A \cap S = A$
12. $A \cap \phi = \phi$
13. $A \cup S = S$
14. $A \cup \phi = A$
15. $A \cup A^c = S$
16. $A \cap A^c = \phi$

Three types of sets we will encounter:

- A set is <u>finite</u> if it contains a finite number of elements, i.e., $A=\{x_1, \ldots, x_n\}$
- A set is <u>countable</u> if its elements can be put into one-to-one correspondence with the integers, i.e., $A=\{x_1, x_2, \ldots\}$
- A set is <u>uncountable</u> if it not finite or countable. A set that is uncountable cannot be written as $\{x_1, x_2, \ldots\}$

  Note: $\mathbb{R}$, and any interval in $\mathbb{R}$, are uncountable. $\mathbb{Q}$ (the set of rationals) is countable.

We will often consider indexed collections of sets, such as

$$\{A_i, i \in I\}$$

where I is called the index set.

The index set can be

- finite, $I = \{1, \ldots, n\}$, so the collection of sets is $\{A_1, \ldots, A_n\}$
- countable, $I = \mathbb{N} = \{1, 2, 3, \ldots\}$, so the collection is $\{A_1, A_2, \ldots\}$
- or uncountable, so the collection is $\{A_\alpha, \alpha \in I\}$ for an uncountable set I. If $I = \mathbb{R}$, for example, then a set in the collection can be written as A_α for some real number α.

Defn. The union of an indexed family of sets is defined as

$$\bigcup_{i \in I} A_i = \{ \omega \in \mathcal{S} : \omega \in A_i \text{ for at least one } i \in I \}$$

Note: If I is finite, can write as

$$\bigcup_{i=1}^{n} A_i$$

~~If~~

If I is countable, can write as

$$\bigcup_{i=1}^{\infty} A_i$$

If I is uncountable, write as

$\bigcup_{i \in I} A_i$, as in definition

Defn. The intersection of an indexed family of sets is defined as

$$\bigcap_{i \in I} A_i = \{ \omega \in \mathcal{S} : \omega \in A_i \text{ for every } i \in I \}$$

Note: Can write as $\bigcap_{i=1}^{n} A_i$ if I is finite, $\bigcap_{i=1}^{\infty} A_i$ if I is countable.

Defn. The collection $\{A_i, i \in I\}$ is <u>disjoint</u> if

$$A_i \cap A_j = \emptyset \quad \forall i, j \in I, \ i \neq j$$

Defn. The collection $\{A_i, i \in I\}$ is a <u>partition</u> of S if it is disjoint and if

$$\bigcup_{i \in I} A_i = S$$