COMER
April 5, 2016

ECE 302 Exam 3

1. Enter your name and signature in the space provided below. YOUR SIGNATURE CERTIFIES THAT YOU WILL NOT ENGAGE IN ANY CHEATING DURING THIS EXAM.
2. You may not use a calculator or any other reference materials.

Name:

SOLUTION

Signature:

No Partial Credit

You must CLEARLY mark your answer for each problem.

1. (5 points) If X is a random variable with probability mass function $p_X(0) = \frac{1}{4}$, $p_X(\frac{1}{2}) = \frac{1}{4}$, and $p_X(\frac{5}{4}) = \frac{1}{2}$, what is $E[X]$?

   (a) 0

   (b) 1/8

   (c) 2/8

   (d) 3/8

   (e) 4/8

   (f) 5/8

   (g) 6/8

   (h) 7/8

   (i) 1

$$E[X] = 0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{5}{4} \cdot \frac{1}{2} = \frac{6}{8}$$

2. (5 points) Given continuous random variables X and Y with joint probability density function $f_{XY}(x, y)$ and joint cumulative distribution function $F_{XY}(x, y)$, all of the following must have the same value EXCEPT:

   (a) $\lim_{y\to\infty} [F_{XY}(b, y) - F_{XY}(a, y)]$ $= F_X(b) - F_X(a) = P(a < X \leq b)$

   (b) $\int_a^b f_X(x)dx$ $= P(X \in (a, b]) = P(a < X \leq b)$

   (c) $\int_a^b \int_{-\infty}^{\infty} f_{XY}(x, y)dydx$ $= \int_a^b f_X(x)dx = P(a < X \leq b)$

   (d) $F_{XY}(b, y) - F_{XY}(a, y)$ $\neq P(a < X \leq b)$

   (e) $P(a < X \leq b)$

3. (5 points) Let X and Y be independent discrete random variables each uniformly distributed between 0 and 3 (i.e., the marginal probability mass functions $p_X(x)$ and $p_Y(y)$ are constant for $x = 0, 1, 2, 3$ and $y = 0, 1, 2, 3$, respectively). What value does the joint probability mass function $p_{XY}(x, y)$ take for every $x = 0, 1, 2, 3$ and $y = 0, 1, 2, 3$?

   (a) 3

   (b) 9

   (c) $\frac{1}{3}$

   (d) $\frac{1}{9}$

   (e) 4

   (f) 16

   (g) $\frac{1}{4}$

   (h) $\frac{1}{16}$

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$
$$= \frac{1}{4} \cdot \frac{1}{4} = 16 \quad \text{for every}$$
$$x = 0, 1, 2, 3; \; y = 0, 1, 2, 3$$

Limited Partial Credit

You must put your final answers in the boxes provided. Credit of either 0, 5, or 10 points will be granted for each problem. You must justify your solution to get credit.

4. (10 points) Two random variables X and Y have joint probability density function

$$f_{XY}(x,y) = \begin{cases} cxe^{-y}, & x \in [0,1], y \in [0,1] \\ 0, & otherwise \end{cases}$$

Find the marginal probability density function of X. You may leave your answer in terms of c.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = c\int_0^1 xe^{-y}dy$$

$$= cx(-e^{-y})\Big|_0^1 = cx(1-e^{-1}),$$

$$x \in [0,1]$$

Answer: $f_X(x) = cx(1-e^{-1}), \quad x \in [0,1]$

5. (10 points) For the previous problem, find the value of c.

$$\int_0^1 cx(1-e^{-1})dx = 1$$

So $c(1-e^{-1})(\frac{1}{2}x^2)\Big|_0^1 = 1$

$$c(1-e^{-1})(\tfrac{1}{2}) = 1$$

$$c = \frac{2}{1-e^{-1}}$$

Answer:

6. (10 points) Consider a random variable X. If $E[X] = 1$ and $Var(X) = 4$, find $E[(3+X)^2]$.

$E[(3+X)^2] = E[9] + E[6X] + E[X^2]$

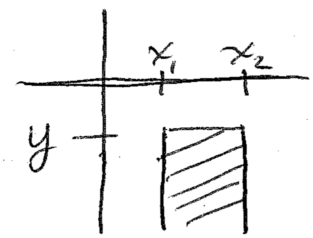
$E[X^2] = Var(X) + \bar{X}^2 = 5$

$E[(3+X)^2] = 9 + 6\bar{X} + 5 = 20$

Answer:
20

7. (10 points) Let X and Y be continuous random variables. For any real numbers x_1, x_2, y with $x_1 < x_2$, write $P(x_1 < X \le x_2, Y \le y)$ in terms of the joint cumulative distribution function of X and Y.

$\{x_1 < X \le x_2, Y \le y\}$ is the shaded region.



Can write

$\{X \le x_2, Y \le y\} = \{X \le x_1, Y \le y\} \cup \{x_1 < X \le x_2, Y \le y\}$

So $\overset{F_X(x_2, y)}{\cancel{P(X \le x_2, Y \le y)}} = F_X(x_1, y) + P(x_1 < X \le x_2, Y \le y)$

$\Rightarrow P(x_1 < X \le x_2, Y \le y) = F_{XY}(x_2, y) - F_{XY}(x_1, y)$

Answer:

Partial Credit Problems

You must justify your solution to get credit. Partial credit will be given at the discretion of the instructor.

8. (20 points) Let X be a discrete uniform random variable with probability mass function $p_X(k) = \frac{1}{7}$ for $k = -3, -2, -1, 0, 1, 2, 3$. If $Y = \max(X, 0)$, find the probability mass function of Y.

$$R_X = \{-3, -2, -1, 0, 1, 2, 3\}$$

$$\Rightarrow R_Y = \{-3, -2, -1, 0\}$$

$$\{Y = -3\} = \{X = -3\}$$

$$\{Y = -2\} = \{X = -2\}$$

$$\{Y = -1\} = \{X = -1\}$$

$$\{Y = 0\} = \bigcup_{i=0}^{3} \{X = i\}$$

So

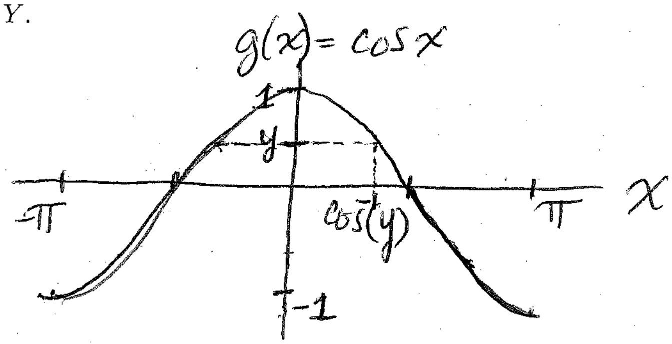
$$p_{\cancel{X}Y}(-3) = p_X(-3) = \frac{1}{7}$$

$$p_Y(-2) = p_X(-2) = \frac{1}{7}$$

$$p_Y(-1) = p_X(-1) = \frac{1}{7}$$

$$p_Y(0) = \sum_{i=0}^{3} p_X(i) = \frac{4}{7}$$

9. (20 points) Let X be a continuous random variable that is uniformly distributed on $[-\pi, \pi]$, so that $f_X(x) = \frac{1}{2\pi}$ for $-\pi \le x \le \pi$ and 0 elsewhere. If $Y = \cos X$, find the cumulative distribution function of Y.



for every $-1 \le y \le 1$, define $\cos^{-1}(y)$ as the solution of $y = \cos x$ in $[0, \pi]$.

for $-1 \le y \le 1$:

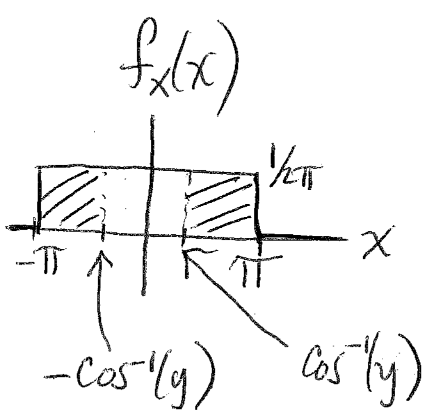
$$F_Y(y) = P(Y \le y) =$$

$$P\left(\{X \in [-\pi, -\cos^{-1}(y)]\} \cup \{X \in [\cos^{-1}(y), \pi]\}\right)$$

$$= P(-\pi \le X \le -\cos^{-1}(y)) + P(\cos^{-1}(y) \le X \le \pi)$$

$$= \frac{1}{2\pi}\left[\left(-\cos^{-1}(y) - (-\pi)\right) + \left(\pi - \cos^{-1}(y)\right)\right]$$

$$= \frac{1}{2\pi}\left[2\left(\pi - \cos^{-1}(y)\right)\right] = 1 - \frac{\cos^{-1}(y)}{\pi}$$



$$F_Y(y) = \begin{cases} 1 - \frac{\cos^{-1}(y)}{\pi}, & -1 \le y \le 1 \\ 0, & y < -1 \\ 1, & y > 1 \end{cases}$$