COMER
March 9, 2015

ECE 302 Exam 2

1. Enter your name and signature in the space provided below. YOUR SIGNATURE CERTIFIES THAT YOU WILL NOT ENGAGE IN ANY CHEATING DURING THIS EXAM.

2. You may not use a calculator or any other reference materials.

Name: SOLUTION

Signature:

No Partial Credit

You must CLEARLY select one answer for each problem.

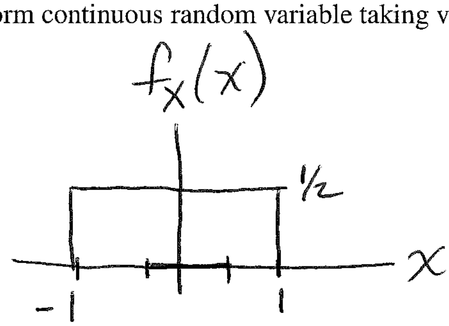
1. (5 points) A student is to answer 7 out of 10 questions on an exam. How many different choices does the student have?

   (a) 7
   (b) 10
   (c) 70
   (d) 120
   (e) 720
   (f) 7000
   (g) 12000

   $\binom{10}{7} = \frac{10!}{3!\,7!} = 120$

2. (5 points) Let X be a uniform continuous random variable taking values in the interval $(-1, 1)$. Find $P(-0.25 < X < 0.5)$.

   (a) 0
   (b) 1
   (c) 1/2
   (d) 3/8
   (e) 3/4
   (f) 1/4
   (g) 5/8
   (h) 2/3



   $P(-.25 < X < .5) = \frac{1}{2}(.5-(-.25)) = \frac{3}{8}$

3. (5 points) For some constant c, the random variable X has probability density function

$$f_X(x) = \begin{cases} cx^4, & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

   Determine the value of c.

   $\int_0^2 cx^4\,dx = 1 \Rightarrow c = \frac{5}{32}$

   (a) 17/18
   (b) 1/4
   (c) 3/8
   (d) 10/17
   (e) 72/79
   (f) 5/32
   (g) 6/11
   (h) 1

4. (5 points) Consider a random variable X defined on a probability space (S, F, P). Which of the following statements about X are necessarily true? Circle all that apply.

   (a) X is a mapping from the sample space S to the real numbers.
   (b) X is a mapping from the event space F to the real numbers.
   (c) X is non-negative
   (d) X has a cumulative distribution function.
   (e) X has a probability mass function.

Limited Partial Credit

You must put your final answers in the boxes provided. Credit of either 0, 5, or 10 points will be granted for each problem. You must show your work to get credit.

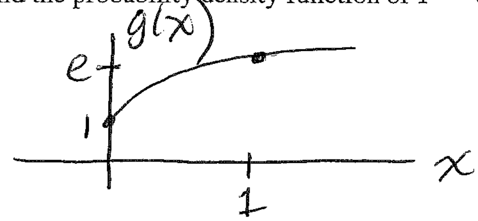
5. (10 points) A committee of 6 people is to be chosen from a group of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible? Two cases: 3W, 3M or 4W, 2M.

First case: For each of the $\binom{8}{3}$ ways to choose women, $\binom{7}{3}$ ways to choose men. Similarly for second case. Total # of committees is the sum of combinations for each of the two cases.

Answer: $\binom{8}{3}\binom{7}{3}+\binom{8}{4}\binom{7}{2}$

6. (10 points) If X is uniformly distributed over $(0,1)$, find the probability density function of $Y = e^X$.

$0 < X < 1 \Rightarrow 1 < Y < e$



For $1 < y < e$,

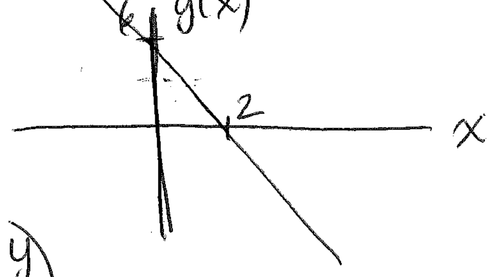
$$F_Y(y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{f_X(\ln y)}{y}$$

Since $f_X(x) = 1$ for $0<x<1$, $f_Y(y) = \frac{1}{y}$ for $1<y<e$

Answer: $f_Y(y) = \begin{cases} 1/y & 1<y<e \\ 0 & \text{elsewhere} \end{cases}$

7. (10 points) Let X be a continuous random variable with probability density function $f_X(x)$. Find the cumulative distribution function of $Y = 6 - 3X$.



For every $y \in \mathbb{R}$,

$$F_Y(y) = P(6-3X \le y)$$

$$= P\left(X \ge \frac{6-y}{3}\right) = 1 - F_X\left(\frac{6-y}{3}\right)$$

Answer: $1 - F_X\left(\frac{6-y}{3}\right) \quad \forall y \in \mathbb{R}$

8. (10 points) For some constant c, the random variable X has probability density function

$$f_X(x) = \begin{cases} cx^n, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find $P(X > x)$ for $0 < x < 1$. Note that your answer should not be in terms of c.

Find c: $\int_0^1 cx^n\,dx = \frac{c}{n+1} = 1 \Rightarrow c = n+1$

So $P(X > x) = \int_x^1 (n+1)t^n\,dt = 1 - x^{n+1}$

Answer: $1 - x^{n+1}$

9. (10 points) Let X be the number of heads in three independent tosses of a fair coin. Find the probability mass function of X.

We have $R_X = \{0,1,2,3\}$, so we need to find $p_X(0)$, $p_X(1)$, $p_X(2)$, $p_X(3)$

$X = 0$ if TTT $\Rightarrow P(X=0) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$

$X = 1$ if HTT, THT, or TTH $\Rightarrow$

$P(X=1) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 3/8$

$X = 2$ if HHT, HTH, or THH $\Rightarrow P(X=2) = \frac{3}{8}$

$X = 3$ if HHH $\Rightarrow P(X=3) = \frac{1}{8}$

Answer: $$p_X(x) = \begin{cases} \frac{1}{8} & x = 0, 3 \\ \frac{3}{8} & x = 1, 2 \end{cases}$$

Partial Credit Problems

You must completely justify your solution to get full credit. Partial credit will be given at the discretion of the instructor.

10. (20 points) Let X be a discrete random variable with a probability mass function given by $p_X(-2) = \frac{1}{12}, p_X(-1) = \frac{1}{6}, p_X(0) = \frac{1}{4}, p_X(1) = \frac{1}{3}$, and $p_X(2) = \frac{1}{6}$. Let the random variable Y be given as $Y = 3X^2 + 1$. Find the probability mass function of Y.

$$R_Y = \{1, 4, 13\}$$

$$P(Y=1) = P(\{X=0\}) = \frac{1}{4}$$

$$P(Y=4) = P(\{X=-1\} \cup \{X=1\}) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$P(Y=13) = P(\{X=-2\} \cup \{X=2\}) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$$

So

$$P_Y(y) = \begin{cases} \frac{1}{4} & y = 1 \\ \frac{1}{2} & y = 4 \\ \frac{1}{4} & y = 13 \end{cases}$$

11. (10 points) Consider a discrete random variable X with range space $R_X = \{1, 2, \ldots\}$. Show that it is not possible for the values that X can take to be equally likely. In other words, show that we cannot have $P(X = i)$ be the same for every $i = 1, 2, \ldots$?

There are two possibilities here: $P(X=i) = 0 \ \forall i$ or $P(X=i) = a > 0$ for some $a \in \mathbb{R}$ (with $a < 1$) $\forall i$

If $P(X=i) = 0 \ \forall i$, then $\sum_{i=1}^{\infty} p_X(i) = 0$, which contradicts the requirement that $\sum_{i=1}^{\infty} p_X(i) = 1$

If $P(X=i) = a > 0 \ \forall i$, then $\sum_{i=1}^{\infty} p_X(i) = \infty$, which also contradicts $\sum_{i=1}^{\infty} p_X(i) = 1$.