COMER
February 9, 2015

ECE 302 Exam 1

1. Enter your name and signature in the space provided below. YOUR SIGNATURE CERTIFIES THAT YOU WILL NOT ENGAGE IN ANY CHEATING DURING THIS EXAM.
2. You may not use a calculator or any other reference materials.

Name: SOLUTION

Signature:

No Partial Credit

You must CLEARLY select one answer for each problem.

1. (5 points) Suppose that A and B are disjoint events in a probability space. If $P(A) = 0.3$ and $P(B) = 0.5$, what is the probability that either A or B occurs?

   (a) 0.3
   (b) 0.4
   (c) 0.5
   (d) 0.6
   (e) 0.7
   (f) 0.8
   (g) 0.9
   (h) 1.0

$P(A \cup B) = P(A) + P(B) = .3 + .5 = .8$

2. (5 points) Suppose that A and B are disjoint events in a probability space. If $P(A) = 0.3$ and $P(B) = 0.5$, what is the probability that A occurs but B does not?

   (a) 0.3
   (b) 0.4
   (c) 0.5
   (d) 0.6
   (e) 0.7
   (f) 0.8
   (g) 0.9
   (h) 1.0

$P(A \cap B^c) = P(A)$ since

$A \cap B = \phi \Rightarrow A \cap B^c = A$

3. (5 points) How many different three-topping pizzas are possible if there are a total of 10 toppings to choose from? Note that a given topping can be selected only once and the ordering of the three toppings does not matter.

   (a) 40
   (b) 60
   (c) 120
   (d) 180
   (e) 720
   (f) 1020
   (g) 2400

$\frac{10!}{3!\,7!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4}}{\cancel{3} \cdot \cancel{2}} = 120$

4. (5 points) A coin is flipped twice. Assuming that the four events in the sample space $S = \{(H, H), (H, T), (T, H), (T, T)\}$ are equally likely, what is the conditional probability that both flips land on heads given that the first flip lands on heads?

   (a) 0
   (b) 1/16
   (c) 1/8

$P(\{(H,H)\} \mid \{(H,H),(H,T)\})$

$= \frac{P(\{(H,H)\} \cap \{(H,H),(H,T)\})}{P(\{(H,H),(H,T)\})}$

$= \frac{P((H,H))}{P(\{(H,H),(H,T)\})} = \frac{1}{2}$

(d) 3/16

(e) 1/4

(f) 5/16

(g) 3/8

(h) 7/16

(i) 1/2

5. (5 points) For the experiment described in the previous problem, what is the conditional probability that both flips land on heads given that at least one flip lands on heads?

(a) 0

(b) 1/12

(c) 1/6

(d) 1/4

(e) 1/3

(f) 5/12

(g) 1/2

(h) 7/12

(i) 2/3

$$P((H,H) \mid \{(H,H),(H,T),(T,H)\}) = \frac{P(\{(H,H)\})}{P(\{(H,H),(H,T),(T,H)\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

6. (5 points) Which of the following statements is true in general for events A and B?

(a) $P(A) = 0$ implies that $A = \emptyset$.

(b) $A = \emptyset$ implies that $P(A) = 0$.

(c) $P(A - B) = P(A) - P(B)$.

(d) $P(A) = P(B)$ implies that $A = B$.

(e) If A and B are disjoint, then A and B are independent.

Limited Partial Credit

You must put your final answers in the boxes provided. Credit of either 0, 5, or 10 points will be granted for each problem. You must show your work to get credit.

7. (10 points) Give an expression for the sample space of an experiment that consists of randomly picking a real number X between 0 and 1, and then randomly picking a real number Y between 0 and X.

Answer: $S = \{(a,b) : a \in \mathbb{R}, b \in \mathbb{R}, a \in (0,1), b \in (0,a)\}$

8. (10 points) Two persons A and B play a coin-tossing game. A tosses the coin first. If the coin comes up heads, A wins and the game is over. Otherwise, B tosses the coin next, and wins if the coin comes up heads. Otherwise, the coin goes back to A, and the process continues until someone wins. The coin tosses are independent of each other. Suppose the coin is fair. What is the probability that A wins the game?

Can write

$\{A \text{ wins}\} = \{H\} \cup \{TTH\} \cup \{TTTTH\} \cup \cdots$

Now $P(k \text{ tails followed by } 1 \text{ head}) =$

$\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{k+1}$ since tosses are independent and coin is fair.

So $P(A \text{ wins}) = \sum_{k=0,2,4,\ldots} P(k \text{ tails followed by } 1 \text{ head})$

$= \sum_{k=0,2,4,\ldots} \left(\frac{1}{2}\right)^{k+1}$. (Note that this is a geometric series that can be solved to get $P(A \text{ wins}) = \frac{2}{3}$.

Answer: $\sum_{k \text{ even}} \left(\frac{1}{2}\right)^{k+1}$

9. (10 points) For the experiment described in the previous problem, what is the probability that A wins the game if the coin has a probability of 0.3 of landing on heads.

In this case, $P(k \text{ tails followed by } 1 \text{ head})$

$= \left(\frac{7}{10}\right)^k\left(\frac{3}{10}\right)$

So

$P(A \text{ wins}) = \sum_{k=0,2,4,\ldots} P(k \text{ tails followed by } 1 \text{ head})$

$= \sum_{k=0,2,4} \left(\frac{3}{10}\right)\left(\frac{7}{10}\right)^k$

(This is a geometric series that can be solved to get

$$P(A \text{ wins}) = \frac{10}{17}$$

Answer: $\frac{3}{10} \sum_{k=0,2,4,\ldots} \left(\frac{7}{10}\right)^k$

10. (10 points) Consider 3 classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen. How many choices are possible?

Choose 3 out of $3n$ without replacement, order ~~doesn't~~ doesn't matter:

$$\frac{(3n)(3n-1)(3n-2)}{3!} = \frac{(3n)(3n-1)(3n-2)(3n-3)\cdots(2)(1)}{3!\,(3n-3)(3n-4)\cdots(2)(1)}$$

$$= \frac{(3n)!}{3!\,(3n-3)!}$$

Answer: $\frac{(3n)!}{3!\,(3n-3)!} = \binom{3n}{3}$

11. (10 points) For the previous problem, how many choices are there in which all 3 students are in the same class?

Choose 3 out of n from class 1

OR 3 out of n from class 2

OR 3 out of n from class 3, so

$$\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3} = \frac{3n!}{3!\,(n-3)!}$$

Answer: $\frac{3n!}{3!\,(n-3)!} = 3\binom{n}{3}$

Partial Credit Problem

You must completely justify your solution to get full credit. Partial credit will be given at the discretion of the instructor.

12. (20 points) Suppose that an experiment is performed N times. For any event E in the event space, let N_E denote the number of times that E occurs, and define the function $f(E) = N_E/N$. Show that the function f satisfies the axioms of probability.

Axiom 1: $P(E) \geq 0 \quad \forall E \in \mathcal{F}$

So we need $f(E) \geq 0 \ \forall E \in \mathcal{F}$

Since both N and N_E must be nonnegative, we have

$$f(E) = \frac{N_E}{N} \geq 0$$

Axiom 2: $P(S) = 1$

Since the outcome must be in S every time the experiment is performed, we have $N_S = N$.

Thus $$f(S) = \frac{N_S}{N} = 1$$

Axiom 3: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \phi$.

We have

$$f(A \cup B) = \frac{N_{A \cup B}}{N}$$

Since A and B are disjoint,

$$N_{A \cup B} = N_A + N_B$$

Thus $$f(A \cup B) = \frac{N_A + N_B}{N} = \frac{N_A}{N} + \frac{N_B}{N} = f(A) + f(B)$$

Axiom 4: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Now $N_{\bigcup_{i=1}^{\infty} A_i} = \sum_{i=1}^{\infty} N_{A_i}$, so

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} N_{A_i} \Big/ N = \sum_{i=1}^{\infty} f(A_i)$$