COMER
May 9, 2015

ECE 302 Final Exam

1. Enter your name and signature in the space provided below. YOUR SIGNATURE CERTIFIES THAT YOU WILL NOT ENGAGE IN ANY CHEATING DURING THIS EXAM.
2. You may not use a calculator or any other reference materials.

Name: SOLUTION

Signature:

No Partial Credit

You must CLEARLY select one answer for each problem.

1. (5 points) Consider the random process $X(t) = At + b$, where A is a random variable with mean μ_A. What is the expected value of $X(1)$?

   (a) b

   (b) μ_A

   (c) $\mu_A t + b$

   (d) $\mu_A t$

   (e) $A + b$

   (f) $\mu_A + b$

2. (5 points) The Strong Law of Large Numbers states that, with probability 1,

   (a) The sample mean of a Gaussian random variable X is equal to the expected value of X.

   (b) The sum of a sequence of iid random variables converges to a uniform random variable.

   (c) The sample mean of a random variable X is equal to the expected value of X.

   (d) The sum of a sequence of iid random variables converges to a Gaussian random variable.

   (e) The sample mean of a random variable X converges to the expected value of X.

3. (5 points) Consider two random variables X and Y with joint probability mass function $p_{XY}(0,0) = 0.4, p_{XY}(0,1) = 0.1, p_{XY}(1,0) = 0.2, p_{XY}(1,1) = 0.3$. Find the correlation $E[XY]$.

   (a) 0.1

   (b) 0.2

   (c) 0.3

   (d) 0.4

   (e) 0.5

   (f) 0.6

   (g) 0.8

   (h) 0.9

   (i) 1.0

4. (5 points) Consider two zero-mean random variables X and Y with $Var(X) = 5$, $Var(Y) = 3$, and $E[(X+Y)^2] = 12$. Find the correlation $E[XY]$.

   (a) 1/4

   (b) 1/2

   (c) 4

   (d) 2

   (e) 16

   (f) 8

   (g) 0

Limited Partial Credit

You must put your final answers in the boxes provided. Credit of either 0, 5, or 10 points will be granted for each problem. You must show your work to get credit.

5. (10 points) Consider a random process $X(t) = \exp(At)$, where A is a continuous random variable. Find the cumulative distribution function of the random variable $X(t)$ in terms of either the cumulative distribution function or the probability density function of A.

$$F_{X(t)}(x) = P(X(t) \le x) = P(e^{At} \le x)$$

$$\cancel{= P(\ln At \le \ln) =}$$

$$= P(At \le \ln x) = P\left(A \le \frac{\ln x}{t}\right)$$

Answer: $F_{X(t)}(x) = F_A\left(\frac{\ln x}{t}\right) = \int_{-\infty}^{\frac{\ln x}{t}} f_A(a)\,da, \; x > 0$

6. (10 points) The joint density function of random variables X and Y is given by

$$f_{XY}(x, y) = e^{-(x+y)}$$

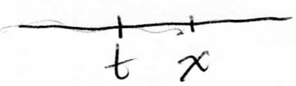
for $x > 0, y > 0$. Find the conditional density function of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{e^{-(x+y)}}{e^{-x}} = e^{-y}, \; y > 0$$

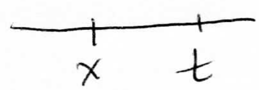
Answer: $e^{-y}, \; y > 0$

7. (10 points) Let X be an exponential random variable with parameter λ. Find the conditional cumulative distribution function $F_X(x|X > t)$ for some real number t. Note that an exponential random variable with parameter λ has density function $f_X(x) = \lambda \exp(-\lambda x)u(x)$.

$$F_X(x|X>t) = P(X \le x | X>t) =$$



$$\frac{P(X \le x, X>t)}{P(X>t)} = \begin{cases} \frac{P(t<X\le x)}{P(X>t)} & x>t \\ 0 & x\le t \end{cases}$$



$$F_X(x|X>t) = \frac{F_X(x)-F_X(t)}{1-F_X(t)}, \quad x>t$$

$$= 1-e^{-\lambda(x-t)}, \quad x>t,\ t>0$$

Answer: $F_X(x|X>t) = 1-e^{-\lambda(x-t)}, \quad x>t>0$

0

8. (10 points) Consider a wide-sense stationary random process $X(t)$ with autocorrelation function $R_X(\tau) = \exp(-|\tau|)$. Find the autocorrelation function of the output $Y(t)$ of the linear time-invariant system with impulse response $h(t) = \exp(-t)u(t)$ when the input to the system is $X(t)$. You may use results derived in class, and you may leave your answer in the form of convolutions.

$$R_Y(\tau) = (\tilde{h} * h * R_X)(\tau)$$

$$\tilde{h}(t) = \exp(t)u(-t)$$

$$R_Y(\tau) = e^{\tau}u(-\tau) * e^{-\tau}u(\tau) * e^{-|\tau|}$$

Or $R_Y(\tau) = \mathcal{F}^{-1}\{|H(\omega)|^2 S_X(\omega)\}$, where $H(\omega) = \mathcal{F}\{e^{-t}u(t)\}$

Answer:

Partial Credit Problems

You must completely justify your solution to get full credit. Partial credit will be given at the discretion of the instructor.

9. (20 points) Let $Z(t) = At^3 + B$, where A and B are independent continuous random variables, each uniformly distributed on $(0, 1)$. Find the mean function $\mu_Z(t)$ and the autocorrelation function $R_{ZZ}(t_1, t_2)$.

$$\mu_Z(t) = E[At^3+B] = \cancel{A\,E[t^3]}$$

$$E[A]t^3 + E[B] = \frac{1}{2}t^3 + \frac{1}{2}$$

$$R_{ZZ}(t_1,t_2) = E[(At_1^3+B)(At_2^3+B)]$$

$$= E[A^2t_1^3t_2^3 + ABt_1^3 + ABt_2^3 + B^2]$$

$$= E[A^2]t_1^3t_2^3 + E[AB]t_1^3 + E[AB]t_2^3 + E[B^2]$$

$$E[AB] = E[A]E[B] = \frac{1}{4}$$

$$Var(A) = \frac{1}{12}$$

$$E[A^2] = Var(\cancel{A}) - (E[A])^2 \Rightarrow$$

$$E[A^2] = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

$$R_{ZZ}(t_1,t_2) = \frac{1}{3}t_1^3t_2^3 + \frac{1}{4}t_1^3 + \frac{1}{4}t_2^3 + \frac{1}{3}$$

10. (20 points) Consider two random variables X and Y with joint probability mass function $p_{XY}(0,0) = 0.4, p_{XY}(0,1) = 0.1, p_{XY}(1,0) = 0.2, p_{XY}(1,1) = 0.3$. Find the conditional probability mass function of X given $Y = 1$.

$$P_{X|Y}(x|1) = \frac{P_{XY}(x,1)}{P_Y(1)}$$

$$P_Y(1) = .1 + .3 = .4$$

$$P_{X|Y}(0|1) = \frac{.1}{.4} = \frac{1}{4}$$

$$P_{X|Y}(1|1) = \frac{.3}{.4} = \frac{3}{4}$$