Conditional Distributions

How do we write conditional probabilities using cdfs, pdfs and pmfs?

Recall that for two events A and B,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

as long as $P(B) \neq 0$.

To write a form of a cdf, consider the case $A = \{X \leq x\}$ for a rv X and some $x \in \mathbb{R}$.

Then we have

$$P\left(X \leq x | B\right) = \frac{P\left(\{X \leq x\} \cap B\right)}{P\left(B\right)}$$

This is called the conditional cdf of X given B and is denoted $F_X(x|B)$. The conditional pdf of X given B is then

$$f_X\left(x|B\right) = \frac{d}{dx}F_X\left(x|B\right)$$

One particular case of interest is

$$B = \{X > t\}$$

for some $t \in \mathbb{R}$.

then we get

$$F_X(x|X > t) = \frac{P\left(\{X \leq x\} \cap \{X > t\}\right)}{P(X > t)}$$

We want to write the right-hand side in terms of the cdf of X, so consider two cases:

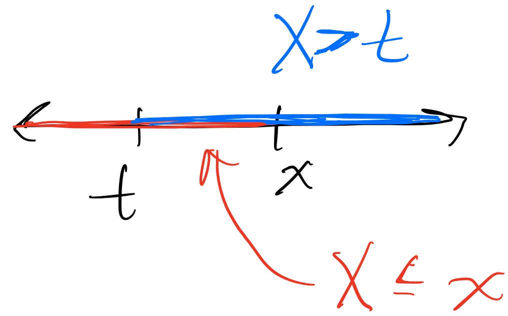
1 $x \leq t$:

The event that X lies in both shaded regions is ϕ, so

$$F_X(x|X > t) = \frac{P\left(\phi\right)}{P(X > t)} = 0$$

for $x \leq t$.

2 $x > t$;



$$\{X \leq x\} \cap \{X > t\} = \{t < X \leq x\}$$

in this case, so

$$F_X(x|X > t) = \frac{P(t < X \leq x)}{P(X > t)}$$

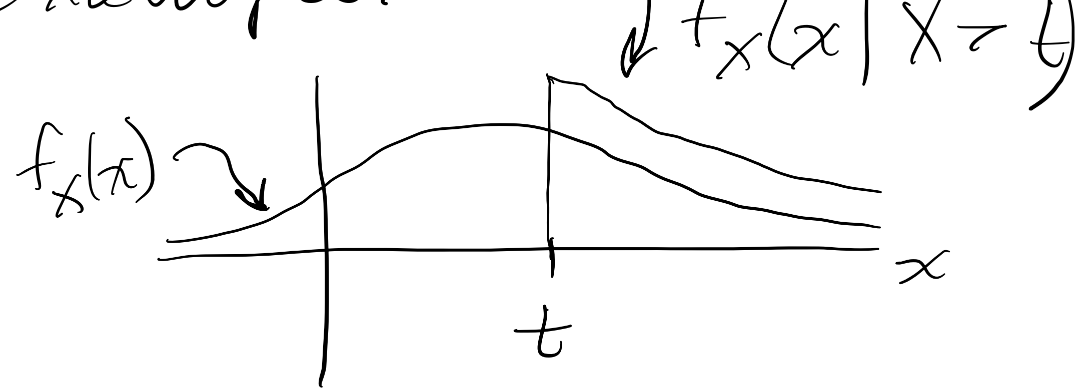
$$= \frac{F_X(x) - F_X(t)}{1 - F_X(t)}$$

Combining the two cases and differentiating wrt x to get the conditional pdf gives

$$f_X(x|X > t) = \begin{cases} 0, x \leq t \\ \dfrac{f_X(x)}{1 - F_X(t)}, x > t \end{cases}$$

This distribution can be used in modeling rare or extreme events in a field called <u>extreme value statistics</u>.

Example.



We have now defined

$$F_X(x|B) = P(X \leq x|B).$$

Using Bayes' Theorem,
we have

$$P(X \leq x | B) = \frac{P(B | X \leq x) P(X \leq x)}{P(B)}$$

assuming $P(X \leq x) \neq 0$
and $P(B) \neq 0$.

In practice we often want to condition on $\{X = x\}$ instead of $\{X \leq x\}$. If X is a continuous rv, we cannot use our previous definition of cond. probability to do this since $P(X = x) = 0 \quad \forall x \in \mathbb{R}$

in that case.

Instead, let

$$P(B|X = x) = \lim_{\Delta x \to 0} P(B|x < X \leq x + \Delta x)$$

Using some Calculus leads to

$$P(B|X = x) = \frac{f_X(x|B)\,P(B)}{f_X(x)}$$

for all x where $f_X(x) \neq 0$.

Now consider $f_X(x|B)$ for the case $B = \{Y = y\}$ for some $y \in \mathbb{R}$.

Let

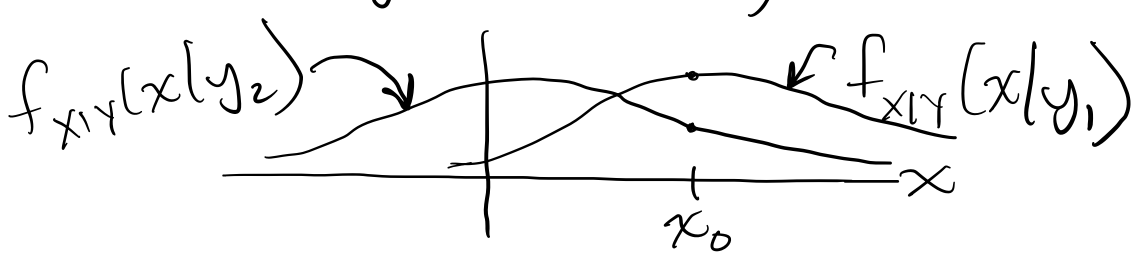
$$f_X(x|Y = y) = \lim_{\Delta y \to 0} f_X(x|y < Y \leq y + \Delta y).$$

Using some more Calculus gives

$$f_X(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

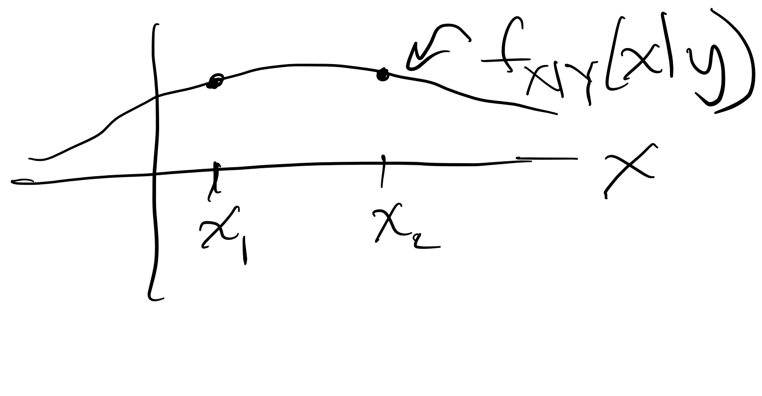
$\forall y$ where $f_Y(y) \neq 0$

Notation: We will write $f_X(x|Y=y)$ as $f_{X|Y}(x|y)$. Note that this is a function of two variables x, y. For two different values y_1, y_2, we might have, for example



So the value of the function $f_{X|Y}(x|y)$ at a point x_0 depends not only on the value of x_0, but also on the value of y.

For a fixed y and two different values of x, the values of the function can be determined with one graph



We have found that

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Similarly, we could find

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Solving these two for f_{XY} and equating them gives

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$$

This is another form of Bayes' Theorem.

We also have another form of the Total Probability Law (TPL):

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_X(x)\, dx$$