Two Functions of Two Random Variables

We are considering the rvs Z and W defined as

$$Z = g(X, Y)$$
$$W = h(X, Y)$$

for two functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ and two rvs X, Y.

Want to find the joint pdf f_{ZW}.

Use a change of variables approach.

First, assume that

- for any fixed $w, z \in \mathbb{R}$, there is a unique

Solution to the system $z = g(x,y)$, $w = h(x,y)$ that will be written as

$$x = g^{-1}(z, w)$$
$$y = h^{-1}(z, w)$$

Note that for the linear transformation example described in the previous lecture, this means that $A_{2\times 2}$ is invertible.

- The partials $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ all exist

Under these two assumptions, it can be shown that

$$f_{ZW}(z,w) = \frac{f_{XY}\left(g^{-1}(z,w), h^{-1}(z,w)\right)}{\left|\dfrac{\partial(z,w)}{\partial(x,y)}\right|}$$

where the notation in the denominator means

$$\left|\frac{\partial(z,w)}{\partial(x,y)}\right| = \left|\det \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix}\right|$$

matrix determinant

$$= \left|\frac{\partial z}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial w}{\partial x}\right|$$

absolute value

Example. Let X and Y be two iid (independent and identically distributed) Gaussian rvs with

$\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = \sigma$, $r = 0$

Now let

$$R = \sqrt{X^2 + Y^2}$$

and

$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Find $f_{R\Theta}$.

In this case

$$g(x, y) = \sqrt{x^2 + y^2}$$
$$h(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

where $-\pi \le \tan^{-1}\left(\frac{y}{x}\right) \le \pi$,

Note that this r is different from the r in the joint the Gaussian pdf

and

$$g^{-1}(r,\theta) = r\cos\theta$$
$$h^{-1}(r,\theta) = r\sin\theta$$

Taking the partial derivatives and computing the Jacobian gives the value $\frac{1}{r}$, which leads to

$$f_{R\Theta}(r,\theta) = \frac{f_{XY}(r\cos\theta, r\sin\theta)}{\frac{1}{r}}$$

$$= \frac{r}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]$$

if $r \geq 0$ and $-\pi \leq \theta \leq \pi$,

and

$-\theta \leq \pi$,

$$f_{R\Theta}(r,\theta) = 0$$

and if $r < 0$ or $\theta \notin [-\pi, \pi]$

Note that the marginal pdf of R can now be found using

$$f_R(r) = \int_{-\pi}^{\pi} f_{R\Theta}(r, \theta)\, d\theta$$

$$= \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{\sigma^2}\right] u(r)$$

This is called a <u>Rayleigh</u> rv. One place it is used is in modeling certain wireless communications channels.

The approach we used to find f_R here can be used to find the pdf of one function of two rvs, $Z = g(X, Y)$. An auxiliary rv W can be defined, f_{ZW} can be found using change of variables formula, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w)\, dw$$

Sometimes the best choice for W is simply $W = X$ or $W = Y$.

Joint Expectation

Given two rvs X and Y, and a rv $Z = g(X, Y)$, we know that

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz$$

from the definition of the expected value of a rv. But it can be shown that

$$E[g(X,Y)] = \int\int_{\mathbb{R}^2} g(x,y) f_{XY}(x,y) dx dy$$

This is often an easier way to find $E[g(X,Y)]$, for a function g, than first finding $f_Z(z)$ ~~for~~

$Z = g(X,Y)$.

For the discrete X and Y case, use

$$E[g(X,Y)] = \sum_{x \in \mathcal{R}_X} \sum_{y \in \mathcal{R}_Y} g(x,y)\, p_{XY}(x,y)$$

Some important special cases:

- $g(x,y) = x$:

$$E[g(X,Y)] = \int\int_{\mathbb{R}^2} x f_{XY}(x,y)\, dx dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy}_{f_X(x)} dx$$

$$= E[X]$$

This is consistent with our original definition of E[X].

- $g(x,y) = (x-\mu_X)^2$

$$E[g(X,Y)] = Var(X)$$

- The <u>correlation</u> between X and Y is

$$Corr(X,Y) = E[XY]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dxdy$$

- The <u>covariance</u> of X and Y is

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

- The <u>correlation coefficient</u> of X and Y is

$$r = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some comments:

- The correlation coefficient is a value commonly used in practice as one measure of the joint behavior of X and Y
- It can be shown that

$$-1 \leq r \leq 1$$

For the cases $r=-1$, $r=1$,

$Y = aX$

for some real number $a < 0$,

$a > 0$, respectively

- If X and Y are independent, then $r = 0$. This can be proved by showing that

$$E\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right]=0$$

or

$$E\left[XY\right]=\mu_X\mu_Y$$

The converse is not necessarily true:

$r = 0 \not\Rightarrow X, Y$ ind.

- It can be shown that the parameter r in the jointly Gaussian pdf is the correlation coefficient, and that for the Gaussian case, $r=0$ does imply X and Y are ind.

Two final definitions:

- If $r=0$, then X and Y are said to be <u>uncorrelated</u>.

  The condition $r=0$ is equivalent to
  - $\mathrm{Cov}(X,Y)=0$
  - $E[XY] = \mu_X \mu_Y$

This can be seen from the definitions of correlation, covariance, and correlation coefficient

- If $E[XY] = 0$, then X and Y are called <u>orthogonal</u>.

This condition is not discussed as much as uncorrelated rvs in practice, but it does have some important uses