One Function of Two Random Variables

We have a function g and a rv Z defined by

$$Z = g(X, Y)$$

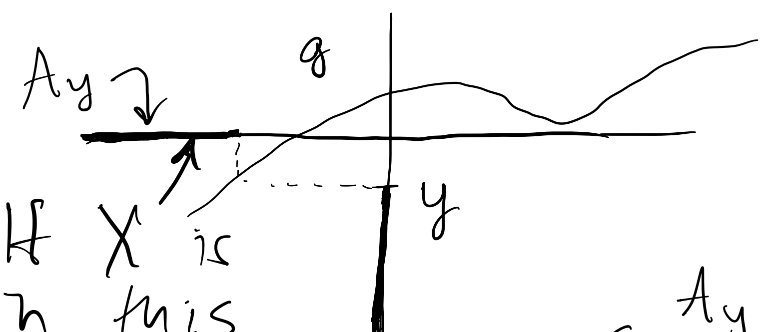
To find $F_Z(z)$, need to find $P(Z \le z)\, \forall z \in \mathbb{R}$.

Basic idea: Need to find $D_z \subset \mathbb{R}^2$ such that

$$\{Z \le z\} = \{(X, Y) \in D_z\}.$$

Must do this for all $z \in \mathbb{R}$.

Recall that for $Y = g(X)$, can use the x-y plane to find the region A_y such that $\{Y \le y\} = \{X \in A_y\}$.



If X is in this interval, then $Y \le y$

Not as easy for $Z = g(X,Y)$ problems. For each $z \in \mathbb{R}$, need to find the region on the x-y plane where the function $g \leq z$, but cannot plot g.

The region D_z is defined as $\{(x,y) \in \mathbb{R}^2 : g(x,y) \leq z\}$

Once D_z is found, use

$$P(Z \leq z) = \int\int_{D_z} f_{xy}(x,y)\,dx\,dy$$

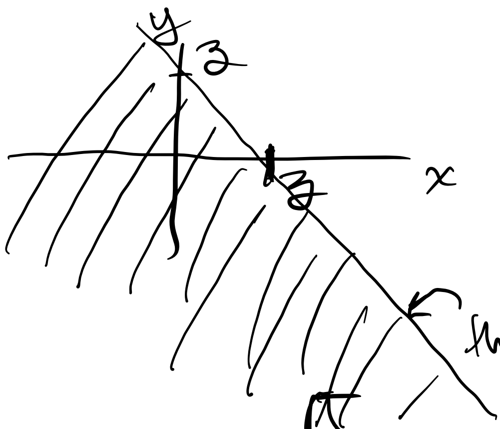
Example. Let $Z = X + Y$.

Then $g(x,y) = x + y$.

(Note that $g(x,y)$ is not random, but $g(X,Y)$ is.)

Now, $\{Z \leq z\} = \{g(X,Y) \leq z\} = \{X + Y \leq z\}$

So $D_z = \{(x,y) \in \mathbb{R}^2 : x + y \leq z\}$



Note: $x+y \leq z$ if $y \leq z-x$

the line $y=z-x$ is the line dividing D_z and D_z^c

So if (X,Y) lies in the shaded region, then $Z \leq z$, or

$$\{(X,Y) \in D_z\} = \{Z \leq z\}$$

Then we can write

$$F_Z(z) = \int\int_{D_z} f_{XY}(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dydx$$

Note that for this example, even though

$Z = X + Y$ (which means that $\forall \omega \in \mathcal{S}, Z(\omega) = X(\omega) + Y(\omega)$), it is not true that $z = x + y$, necessarily. This is only true for points (x, y) on the line $y = z - x$.

Three challenges for $Z = g(X, Y)$ problems:

1. For which values of z can I use the same solution. For example, will the form of the integration be different for $z < 0$ and $z \geq 0$.
2. What is D_z for each $z \in \mathbb{R}$?
3. What are the endpoints

for integrating over D_z?

Continuing the example $Z = X+Y$, consider the case where X, Y are independent. Then

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x)\, f_Y(y)\, dy dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \underbrace{\int_{-\infty}^{z-x} f_Y(y)\, dy}dx$$

(Integrate the pdf to get the cdf)

So $$F_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, F_Y(z-x)\, dx$$

(Recall: $\int_{-\infty}^{y} f_Y(t)\, dt = F_Y(y)$)

To find the pdf of Z:

$$f_Z(z) = \frac{d}{dz}\left[\int_{-\infty}^{\infty} f_X(x)\, F_Y(z-x)\, dx\right]$$

$$= \int_{-\infty}^{\infty} f_X(x) \, \frac{d}{dz} F_Y(z-x) \, dx$$

(differentiate the cdf to get the pdf)

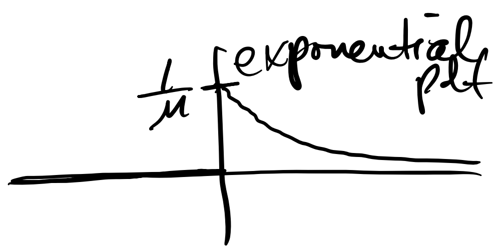
$$= \int_{-\infty}^{\infty} f_X(x) \, f_Y(z-x) \, dx$$

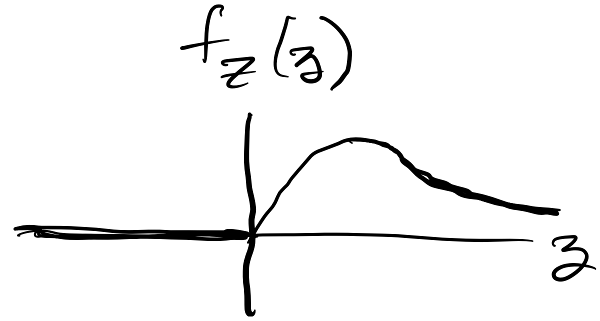
$$= (f_X * f_Y)(z) \; \forall z \in \mathbb{R}$$

So the density function of the sum of independent rvs is <u>not</u> the sum of the density functions.

Example. Let X and Y be independent exponential rvs with $\mu_X = \mu_Y = \mu$. Then if $Z = X + Y$,

$$f_Z(z) = \frac{z}{\mu^2} \exp\left(-\frac{z}{\mu}\right) u(z)$$





Two Functions of Two Random Variables

Consider the two functions $g: \mathbb{R}^2 \to \mathbb{R}$ and $h: \mathbb{R}^2 \to \mathbb{R}$, and two rvs X and Y. Let

$$Z = g(X, Y) \quad \text{and}$$
$$W = h(X, Y).$$

Find the joint distribution of Z and W.

Example. Consider a linear transformation

$$\begin{bmatrix} Z \\ W \end{bmatrix} = A_{2\times 2} \begin{bmatrix} X \\ Y \end{bmatrix}$$

2×2 matrix of elements $a_{ij} \in \mathbb{R}$

Then
$$g(x, y) = a_{11}x + a_{12}y$$
$$h(x, y) = a_{21}x + a_{22}y$$

Theoretically, could find $F_{ZW}(z,w)$ using

$$F_{ZW}(z,w) = \iint_{D_{zw}} f_{XY}(x,y)\,dx\,dy$$

for some $D_{zw} \subset \mathbb{R}^2$

but this is very difficult in practice.