Jointly Gaussian Random Variables

Defn. Random variables X and Y are jointly Gaussian if

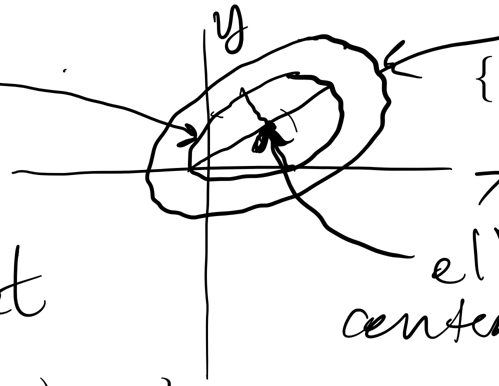
$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}}\cdot$$

$$\exp\left[-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2r(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right]$$

where $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ are the means, variances of X, Y; $r \in \mathbb{R}$, $-1 < r < 1$.

Comments about the jointly Gaussian case:

- Contours of f_{XY} are ellipses



$a_1, a_2 > 0$; $a_1 > a_2$

The major and minor axes lengths and orientational depend on σ_X, σ_Y, r

Note: You do not need to know these details about the contours of f_{XY}, but you might find them insightful.

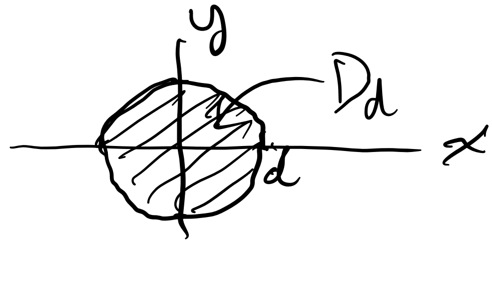
- It can be shown that if X, Y are jointly Gaussian, then each is marginally Gaussian, with

$X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$

However, the converse is not necessarly true:

X, Y each Gaussian $\not\Rightarrow$ X, Y jointly Gaussian

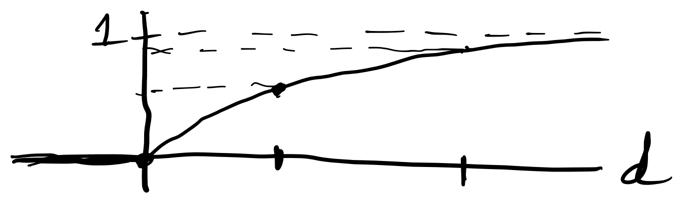
Example. Let X and Y be jointly Gaussian with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = \sigma$, and $r = 0$. Find the probability that (X, Y) lies inside a circle of radius d about the origin.



$$P\left((X,Y) \in D_d\right) = \int\int_{D_d} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) dxdy$$

Using a change of variables to polar coords, $r = x^2+y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ gives

$$P\left((X,Y) \in D_d\right) = 1 - e^{-d^2/2\sigma^2}$$



Note that this problem could have been phrased:

What is the probability:

- $P(X^2+Y^2 \leq d^2)$, or
- $P(X^2 \leq d^2 - Y^2)$, or ...

You may need to do some rearranging to get the form $P((X,Y) \in D)$

Statistical Independence of Two Random Variables

Defn. Two rvs X, Y are <u>independent</u> if

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$\forall x, y \in \mathbb{R}$$

Equivalently:

- X and Y are ind. if

$$F_{XY}(x,y) = F_X(x)\, F_Y(y)$$

$$\forall x, y \in \mathbb{R}$$

- X and Y are ind. if

$$P(X \in A, Y \in B) =$$

$$P(X \in A)\, P(Y \in B),$$

$$A, B \subset \mathbb{R}$$

Proof omitted

- If X and Y are discrete, then they are independent if

$$p_{XY}(x,y) = p_X(x)\,p_Y(y)$$

$$\forall x \in \mathcal{R}_X, y \in \mathcal{R}_Y$$

Proof omitted

One Function of Two Random Variables

Given rvs X and Y, and a function $g: \mathbb{R}^2 \to \mathbb{R}$, let $Z = g(X,Y)$. What is the prob. distribution of Z?
To find $F_Z(z)$, consider

$\{Z \le z\}$ for $z \in \mathbb{R}$.