We have introduced two rvs X and Y and discussed events of the type $\{(X,Y)\in D\}$ for $D\subset\mathbb{R}^2$

Can write probabilities for this type of event in terms of the measure P:

$$P((X,Y)\in D)$$

Now consider more practical representations of the distribution describing X,Y.

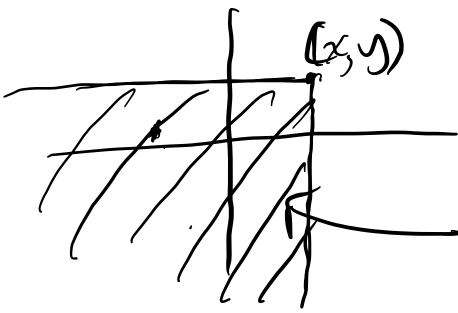
The Joint Cumulative Distribution Function

Defn. The <u>joint cumulative distribution function</u>, or joint distribution function, or joint cdf, of rvs X and Y is

$$F_{XY}(x,y) = P\left(\{X\leq x\}\cap\{Y\leq y\}\right)$$

$= P(X \leq x, Y \leq y) \quad \forall x, y \in \mathbb{R}$

(or all $(x,y) \in \mathbb{R}^2$)



$F_{XY}(x,y) = P((X,Y)$ lies in the shaded region$)$

Some properties of F_{XY}:

1. $$\lim_{x \to -\infty} F_{XY}(x,y) = \lim_{y \to -\infty} F_{XY}(x,y) = 0$$

2. $$\lim_{x \to \infty} F_{XY}(x,y) = F_Y(y), \forall y \in \mathbb{R}$$

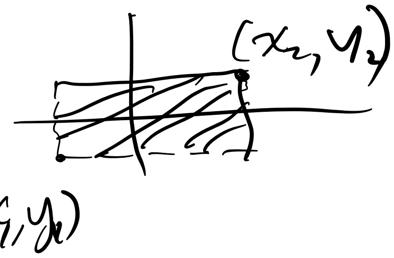
$$\lim_{y \to \infty} F_{XY}(x,y) = F_X(x), \forall x \in \mathbb{R}$$

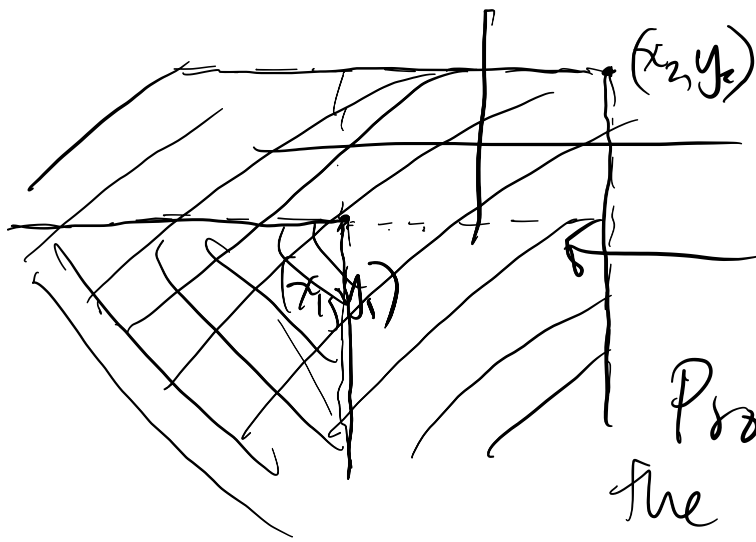
F_X and F_Y are called the <u>marginal</u> cdfs of X and Y

③ For $x_1 < x_2, y_1 < y_2,$

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) =$$

$$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$$





$P(X \le x_2, Y \le y_2)$

Proof left to the student

It is difficult to write $P((X, Y) \in D)$ for some D, where D is a circle for example, in terms of the joint cdf. So the joint pdf is more useful in practice, often.

The Joint Probability Density Function

Defn. The <u>joint probability density function</u>, or joint pdf, or joint density function, of X and Y is

$$f_{XY}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_{XY}(x,y)$$

for all x, y where these partials exist.

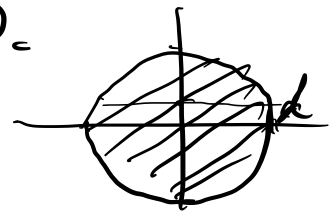
Using Calculus, can show that for $D \subset \mathbb{R}^2$,

$$\boxed{P((X,Y)\in D) = \iint_D f_{XY}(x,y)\,dx\,dy}$$

Important!

Ex. let $D_d = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le d^2\}$

for some $d > 0$.



Then $P((X,Y) \in D_d\}$

$$= \int_{-d}^{d} \int_{-\sqrt{d^2-x^2}}^{\sqrt{d^2-x^2}} f_{XY}(r,s)\, ds dr$$

(would probably want to do a change of variables to polar coordinates to evaluate this)

Some properties of the joint pdf

1. $$\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx dy}_{P\left((X,Y) \in \mathbb{R}^2\right)} = 1$$

2. $F_{XY}(x,y) =$

$$\underbrace{\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(r,s)\,ds\,dr}_{P(X \le x, Y \le y)}, \quad \text{for all } x, y \in \mathbb{R}$$

③ The marginal pdfs of X and Y can be found using

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

If X and Y are both discrete, we use the joint pmf:

The Joint Probability Mass Function

Defn. The joint probability mass function, or joint pmf, of two discrete rvs X, Y is

$$p_{XY}(x,y) = P(\{X=x\} \cap \{Y=y\})$$

$$= P(X=x, Y=y)$$

$\forall x \in R_X, y \in R_Y$

Some properties

① $\sum_{x \in R_X} \sum_{y \in R_Y} p_{XY}(x,y) = 1$

② Marginal pmfs:

$p_X(x) = \sum_{y \in R_Y} p_{XY}(x,y), \ \forall x \in R_X$

$p_Y(y) = \sum_{x \in R_X} p_{XY}(x,y),$

$\forall y \in R_Y$