We have defined the ~~expected~~ expected value $E[X]$ of a rv $X$. We now generalize this concept:

Consider a function $g: \mathbb{R} \to \mathbb{R}$, and let $Y = g(X)$. We know that

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

It can be shown that

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Proof omitted (Calculus)

If you do not need to know $f_Y(y)$, use this form for $E[g(X)]$

Similarly, if $X$ is discrete, use

$$E[Y] = E[g(X)] = \sum_{x \in \mathcal{R}_X} g(x)\, p_X(x)$$

Now consider the special case

$$g(x) = \left(x - \overline{X}\right)^2$$

$\overline{X} \leftarrow E[X]$

Then

$$E\left[g(X)\right] = E\left[\left(X - \overline{X}\right)^2\right]$$

This is the <u>variance</u> of X. Its positive square root

$\sqrt{E[(X-\overline{X})^2]}$ is the

<u>standard deviation</u>

Linearity of Expectation

For two functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, and two real-valued constants (non-random) $a, b \in \mathbb{R}$,

$$E\left[a g_1(X) + b g_2(X)\right] = a E\left[g_1(X)\right] + b E\left[g_2(X)\right]$$

So expectation is a linear operator. The proof of this follows directly from the linearity of integration.

Example.

$$E[a\cos\Theta + b\sin\Theta] = aE[\cos\Theta] + bE[\sin\Theta]$$

if $\Theta$ is a rv and $a, b \in \mathbb{R}$

Example.

Let $X$ be Gaussian with mean $\mu$ and variance $\sigma^2$. (Note: It can be shown that the $\sigma^2$ parameter in the Gaussian pdf is the variance.)

Find $E[X + X^2]$.

Using the form for $E[g(X)]$ gives

$$E[X + X^2] = \int_{-\infty}^{\infty} (x + x^2) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

Instead of evaluating this

integral, not that due to linearity

$$E\left[X+X^2\right]=E\left[X\right]+E\left[X^2\right]$$

Now can use the results:

- $E[X]=\mu$ from the Gaussian pdf (can be shown using integration by parts)
- $Var(X)=\sigma^2$ from the Gaussian pdf.

  Variance of X

  (Can be shown using integration by parts twice)
- In general, for a rv X,

$$Var\left(X\right)=E\left[X^2\right]-\left(E\left[X\right]\right)^2$$

Proof left to the

Student

to get the result for this example:

$$E\left[X+X^2\right]=\mu+\sigma^2+\mu^2$$

This is much easier than evaluating the integral $E[g(X)]$ in this case

The general result

$$Var\left(X\right)=E\left[X^2\right]-\left(E\left[X\right]\right)^2$$

can be very useful, because it's often easier than finding the integral

$$\int_{-\infty}^{\infty}\left(x-E[X]\right)^2 f_X\left(x\right)dx$$

To prove this result, it is helpful to use:

- $E[a]=a$ for any

constant $a \in \mathbb{R}$, and

- linearity of expectation

Some key points to remember for expectation:

- Definition

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- For discrete rv $X$, use

$$E[X] = \sum_{x \in \mathcal{R}_X} x p_X(x)$$

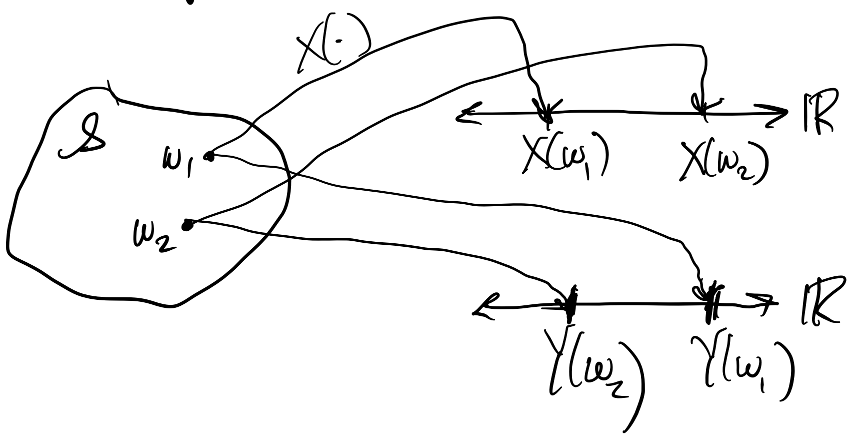
- For $E[g(X)]$, use

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

- Use linearity of $E[\,]$ operator when possible

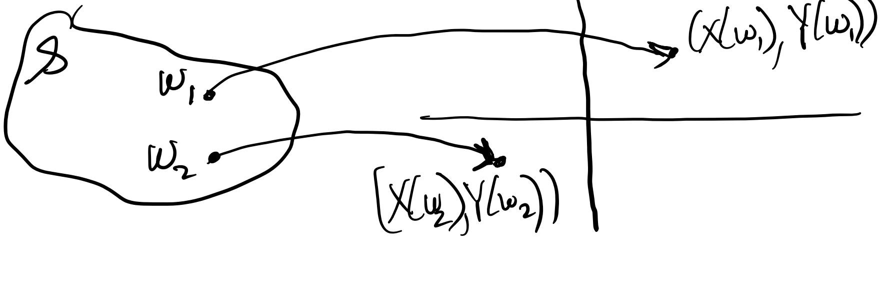
# Two Random Variables

Consider two rvs $X$ and $Y$ defined on a prob. space $(\mathcal{S}, \mathcal{F}, P)$. We could view these as two separate functions on $\mathcal{S}$:




But this is not helpful for characterizing the joint behavior of $X$ and $Y$.

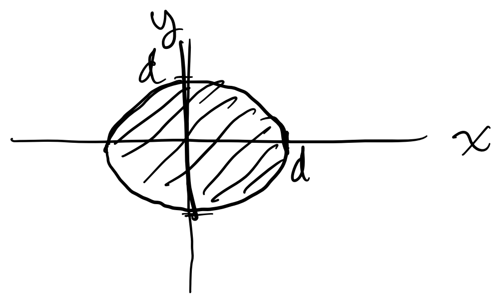
Instead, consider the values that the pair $(X, Y)$ can take, in the $x$-$y$ plane



Events of interest for two rvs will have the form $\{(X,Y) \in D\}$, where $D \subset \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Some examples we will encounter:

- $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le d^2 \text{ for some } d \in \mathbb{R}, d > 0\}$




- $D = \{(x,y) \in \mathbb{R}^2 : x > y\}$