Case 3 : Now consider $Y = g(X)$ when X and Y are both discrete.

First find $\mathcal{R}_Y$. Then for every $y \in \mathcal{R}_Y$, find $A \subset \mathcal{R}_X$ such that

$$\{Y = y\} = \{X \in A_y\}$$

Then we have

$$p_Y(y) = \sum_{x \in A_y} p_X(x) = \sum_{x \in \mathcal{R}_X : g(x) = y} p_X(x)$$

This is true because

$$\{Y = y\} = \bigcup_{x \in A_y} \{X = x\}$$

disjoint events

Ex. Let X be the value rolled on a die, and let

$$Y = \begin{cases} 1 & \text{if } X \text{ is even} \\ 0 & \text{if } X \text{ is odd} \end{cases}$$

Find $p_Y(y)$.

$R_X = \{1, 2, \dots, 6\}$; $R_Y = \{0, 1\}$

Here $g(x) = x \% 2$ and

$$\{Y = 1\} = \{X\%2 = 0\} = \{X \in \{2, 4, 6\}\}$$

So

$$p_Y(1) = \sum_{x \in \{2,4,6\}} p_X(x) =$$

$$p_X(2) + p_X(4) + p_X(6)$$

and

$$p_Y(1) = 1 - p_Y(0) = p_X(1) + p_X(3) + p_X(5)$$

Summary of how to find the distribution of $Y = g(X)$

- Decide if Y is continuous or discrete
  - Plotting $g(x)$ can help determine this

- If you are asked for the pmf of Y, Y is discrete
- If X is discrete, Y is discrete

- If Y is discrete,
  - Find R_Y using the function g
  - Then for each $y \in R_Y$ find A_y such that

$$\{Y = y\} = \{X \in A_y\}$$

where

$A_y \subset \mathbb{R}$ if X is continuous

$A_y \subset R_X$ if X is discrete

  - Find $p_Y(y)$:

$$p_Y(y) = \begin{cases} \int_{A_y} f_X(x)\,dx & \text{if } X \text{ is continuous} \\ \sum_{A_y} p_X(x) & \text{if } X \text{ is discrete} \end{cases}$$

- If Y is continuous,
  - For every $y \in \mathbb{R}$, find $A_y \subseteq \mathbb{R}$ such that $\{Y \leq y\} = \{X \in A_y\}$
  - Use $F_Y(y) = \int_{A_y} f_X(x)\,dx$
  - differentiate wrt y if $f_Y(y)$ is requested
  - or instead, use the change of variables formula, but be careful!

It is a good idea, although not required for full credit, to plot g first.

Expectation

We sometimes characterize a rv using a single parameter, or small set of parameters, instead of the entire distribution. We can use the "expectation" operation for this.

Defn. The <u>expected value</u> of a rv X is

$$E[X] \equiv \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Comments

- $E[X] \in \mathbb{R}$, so it is not random
- Using the sifting property of the δ-function it can be shown that if X is discrete, then

$$E[X] = \sum_{x \in \mathcal{R}_X} x p_X(x)$$

Use this ~~form~~ instead of the defn. of $E[X]$ in terms of the density function, IF X is discrete

- $E[X]$ is also called

the <u>mean</u> of X, and denoted

$E(X)$, EX, $\bar{X}$, μ_X, μ

Note: In statistics the mean is the average of a set of samples. In prob. theory, that average is called the <u>sample mean</u>.

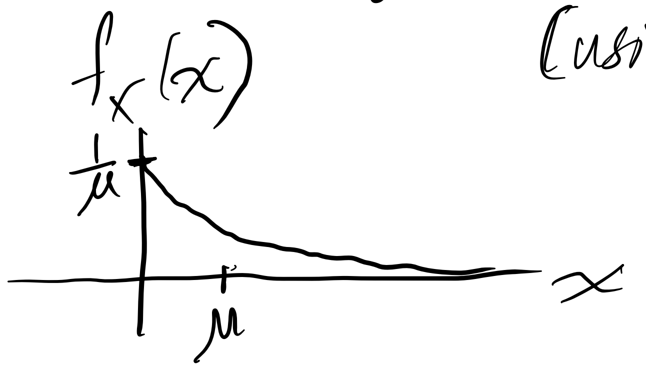
Example. Let X be an exponential rv having

$$f_X(x) = \lambda e^{-\lambda x} u(x).$$

Then $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

(using integration by parts)



For a Gaussian X it can be shown that in the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)}{2\sigma^2}}$$

μ is $E[X]$.