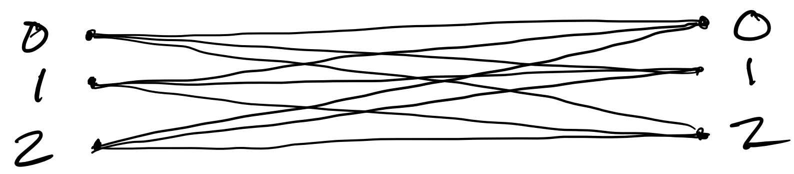
Bayes' Theorem/TPL Example:

Consider a communications system with a 3-symbol alphabet, $\{0,1,2\}$.

<u>Source</u> <u>Channel</u> <u>Receiver</u>



Want to know the probabilities that if the receiver gets a j, for $j=0,1,2$, then the source sent i, for $i=0,1,2$.

Let $S = \{(i,j) : i \text{ is sent}, j \text{ is received, for } i=0,1,2 \text{ and } j=0,1,2\}$

The event that i is sent is $\{(i,0),(i,1),(i,2)\}$

The event that j is received is $\{(0,j),(1,j),(2,j)\}$

To make things more concrete, consider the particular events

$A = \{(0,0), (0,1), (0,2)\}$ — 0 sent

$B = \{(0,0), (1,0), (2,0)\}$ — 0 received

The receiver wants to know $P(A|B)$.

Write as $P(0s|0r)$

0 sent   0 recieved

It can be very difficult to model (or learn) this prob. in practice. Instead use Bayes' Theorem:

$$P(0s|0r) = \frac{P(0r|0s)P(0s)}{P(0r)}$$

- $P(0r|0s)$ can be learned using "channel modeling"

  For each pair i,j, estimate

$$e_{ij} = P(j r \mid i s)$$

- $P(0 s)$ can be learned using "source modeling"
  Estimate $P(i s)$ for $i = 0,1,2$
- $P(0 r)$
  Use the TPL:

$$\boxed{P(0 r) = \sum_{i=0}^{2} P(0r \mid A_i) P(A_i)}$$

where $A_i = \{i \text{ was sent}\}$, $i = 0,1,2$

So $$P(0s \mid 0r) = \frac{P(0r \mid 0s) P(0s)}{P(0r \mid 0s) P(0s) + P(0r \mid 1s) P(1s) + P(0r \mid 2s) P(2s)}$$

Note that:

- $P(0s \mid 0r)$ has been written in terms that can be estimated using source or channel modeling
- A_1, A_2, A_3 form a partition of S so the TPL can be used

Independent Events

Defn. Events A and B are <u>statistically independent</u> if

$$P(A\cap B) = P(A)P(B)$$

In general, $P(A\cap B) \neq P(A)P(B)$, so do not use this equality unless it is given in the prob. statement that A, B are ind., or you have shown that $P(A\cap B) = P(A)P(B)$.

Note that if A, B are ind., then

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)}$$

$$= P(A), \text{ } \underline{\underline{\text{if}}} \text{ } P(B)\neq 0$$

Similarly, $P(B|A) = P(B)$ if A, B are ind. and $P(A)\neq 0$

Conversely, if $P(A)\neq 0$ and $P(B)\neq 0$, <u>and</u> if

$P(A|B)=P(A)$ and $P(B|A)=P(B)$, then A and B are ind.

- You cannot use this to show independence if $P(A)=0$ or $P(B)=0$.

You do not need to know the following definition. The point of it is just to show you how difficult it is to show that n events are independent unless n is quite small!

Defn. Events $A_1, \dots, A_n$ are independent if for every $k=2,\dots,n$, and every $1 \le j_1, \dots, j_k \le n$,

$$P\left(\bigcap_{i=1}^{k} A_{j_i}\right) = \prod_{i=1}^{k} P(A_{j_i}),$$

So for any subcollection of any number k of the n events, this equality must hold for $A_1, \dots, A_n$ to be ind.

Bernoulli Trials—Combinatorics

Consider a random experiment $(S_0, \mathcal{F}_0, P_0)$, with one particular event A of interest. If we assume that n independent trials of the experiment are run, and we know $p = P_0(A)$, what is the prob. that A occurs k times in the n trials?

Ex. A binary source generates an independent sequence of n bits. What is the prob. that k of the bits are 0?