Conditional Probability
(cont'd from Jan 24)

Recall that if $P(B) > 0$, then $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $\forall A \in \mathcal{F}$

To give some intuition using a specific example:

Roll a fair die. Then

Let $A = \{\text{outcome is even}\}$
$= \{2, 4, 6\}$

Let $B = \{\text{outcome} > 3\} =$
$\{4, 5, 6\}$

What is the prob. of A given B?

Given that B occurs, A also occurs if outcome is 4 or 6, or if $A \cap B = \{4, 6\}$ occurs

Since B has 3 outcomes, divide by 3 to get

$P(A|B) = \frac{2}{3}$ ← # of outcomes in B also in A

3 ← # of outcomes in B

$$P(A|B) = \frac{2}{3}$$

$2 \leftarrow$ # of outcomes in B also in A

$3 \leftarrow$ # of outcomes in B

This example illustrates the intuition behind the definition of $P(A|B)$ for the specific case where A, B are finite and P is the counting probability, but

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

can be used for any S, A, B, and P, as long as $P(B) \neq 0$.

Now return to the blood test example: $A = \{\text{person has disease}\}$
$B = \{\text{person tests positive}\}$

Q2: What is the prob. that a person who tests positive has the disease?
This is $P(A|B)$.

Q3: What is the prob that a person who has the disease tests positive?
this is $P(B|A)$.

Comments:

- In translating English to a conditional prob., look for words like
  - assuming that...
  - given that...
  - if ...
  - conditioned on the event...

The ... represents an

event to be conditioned on

- In the phrase "What is the prob that ... given that some event B occurs," the ... is an event A for which you are considering $P(A|B)$
- For any $B \in \mathcal{F}$ with $P(B) \neq 0$, $\underline{P(\cdot|B)}$ is $\underline{\text{a valid prob. measure}}$ $\underline{\text{if } P(\cdot) \text{ is a valid}}$ $\underline{\text{prob. measure}}$, and $P(\cdot|B), P(\cdot)$ are defined on the same $S, \mathcal{F}$, so

Proof omitted

for example: • $P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B)$,

and • $P(A|B) = 1 - P(A^c|B)$

for any $A, B, C \in \mathcal{F}$ with $P(B) \neq 0$

Bayes' Theorem

From the defn. of cond. prob., we have two expressions for $P(A \cap B)$:

$P(A \cap B) = P(A|B)P(B)$, and

$P(A \cap B) = P(B|A)P(A)$

assuming that $P(A) > 0$ and $P(B) > 0$, so

$P(A|B)P(B) = P(B|A)P(A)$,

or $\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$.

This is the first form of Bayes' Theorem we will encounter

- Note that $P(A \cap B) = P(B|A)P(A)$ is not Bayes' Theorem. This is just a restatement of the defn. of cond. prob.
- Sometimes we know $P(B|A)$, $P(A)$, $P(B|A^c)$, but not $P(B)$. How can we find $P(B)$?

The Total Probability Law

Let $A_1, \dots, A_n$ form a partition of S, meaning each outcome is in exactly one A_i, or $\bigcup_{i=1}^{n} A_i = S$, and

$$A_i \cap A_j = \emptyset \quad \forall i \neq j$$

then the Total prob. Law (TPL) states that

$$P(B) = \sum_{i=1}^{n} P(B|A_i) P(A_i), \text{ if}$$

$P(A_i) > 0$.

Proof. $P(B) = P(B \cap S) =$ ← set algebra

$P(B \cap (\bigcup_{i=1}^{n} A_i))$ ← partition property

$= P(\bigcup_{i=1}^{n} (B \cap A_i))$

(other steps to be explained by the student)

So $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$

$= \sum_{i=1}^{n} P(B|A_i) P(A_i)$

Combining Bayes' and the

TPL given

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

for the case $n = 2$
with $A_1 = A$, $A_2 = A^c$

Example using the TPL:
Experiment. Have a full, well-shuffled deck. Set aside the first card without looking at it. Deal the second card. What is the prob that the second card is an ace? Consider the event
$B = \{$second card is an ace$\}$.

Then we can use the partition $A_1 = \{$first card is an ace$\}$, and

A_1^c, and the TPL says

$$P(B) = P(B|A_1)P(A_1) + P(B|A_1^c)P(A_1^c)$$
$$= \left(\frac{3}{51}\right)\left(\frac{4}{52}\right) + \left(\frac{4}{51}\right)\left(\frac{48}{52}\right)$$

So the prob. that the second card is an ace, if we do not know what the first card was, is

$$\frac{12 + (4)(48)}{(51)(52)}$$