# The Probability Measure
# (cont'd from Jan 19)

## Some Properties of P Derived from the Axioms

In previous lecture, showed that ① $P(\phi) = 0$ for any $(S, \mathcal{A}, P)$

But remember —

if $P(A) = 0$, that does not necessarily mean $A = \phi$

(2) $P(A^c) = 1 - P(A) \quad \forall A \in \mathcal{F}$

Proof. Write as $P(A)+P(A^c)=1$.

Then, since $\boxed{A \cap A^c = \emptyset}$, $\leftarrow$ set algebra

$\boxed{P(A \cup A^c) = P(A) + P(A^c)}$, $\leftarrow$ Axiom 3

from Axiom 3.

But $\boxed{A \cup A^c = S}$, $\leftarrow$ Set algebra

so $\boxed{P(S) = P(A) + P(A^c)}$. $\leftarrow$ Substitution of $A \cup A^c$ with $S$ (since if $A=B$ then $P(A)=P(B)$)

Since $\boxed{P(S) = 1}$ $\leftarrow$ Axiom 1

$1 = P(A) + P(A^c)$

Comments:

- started the proof by rewriting $P(A^c) = 1 - P(A)$ as $\underbrace{P(A) + P(A^c) = 1}$
  Can use Axiom 3 in this form
- Be careful with the set difference
  $A - B = A \cap B^c$
  Do not write

not true

$$P(A-B) = P(A) - P(B)$$

set operation

arithmetic operation

in general

$$P(A \cap B^c) \neq P(A) - P(B)$$

Common mistake:

Confusing set and arithmetic operations.

Do not write, e.g.

not valid

$\overline{P(A)}$, $P(A) \cup P(B)$

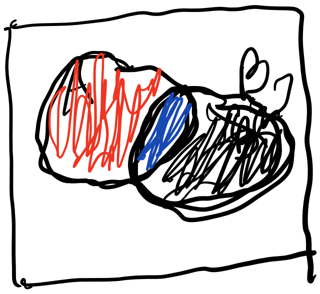
- Complements and unions (and intersection) are set operations; $P(A)$, $P(B)$ are not sets

③ For any $A, B \in \mathcal{F}$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof. Can rearrange as

$$P(A) = P(A \cap B) + P(B) - P(A \cup B)$$




$A \cap B^c$

$A \cap B$

Then we can write $A$ as

$$\boxed{A = (A \cap B) \cup (A \cap B^c)}$$

← Proof left to the student. (Venn diagram is not a proof for general $A, B$)

But $A \cap B$ and $A \cap B^c$ are disjoint,

since $(A \cap B) \cap (A \cap B^c)$

$$= A \cap A \cap \underbrace{B \cap B^c}_{\emptyset} = \emptyset.$$

So $P(A) = P(A \cap B) + P(A \cap B^c)$ (*)

by Axiom 3.

Also, can write

$$A \cup B = (A \cap B^c) \cup B$$

then $P(A \cup B) = P(A \cap B^c) + P(B)$, by Axiom 3 again

Note correction

Substituting into (*)

$P(A \cup B) - P(B)$

$$P(A) = P(A \cap B) + \cancel{P(B) - P(A \cap B^c)}$$

Rearranging gives the result we want

4 If $A \subset B$, then $P(A) \leq P(B)$ $\forall A, B \in \mathcal{F}$.

Proof left to the student

Note that many more properties of $P$ can be proved. In addition to the axioms, these properties are guaranteed to hold for any valid $P$.

This ends our lectures on the prob. space $(S, \mathcal{F}, P)$.

# Conditional Probability

Motivating example.
Consider a blood test for a certain disease.
Three questions we might ask:

1. What is the prob. that a person has the disease and tests positive?
2. What is the prob that a person who tests positive actually has the disease?
3. What is ~~the~~ prob. that a person who has the disease tests positive?

It will turn out that Q2 and Q3 are asking

~~for~~ conditional probs.

To write these probs mathematically, first let

$$\cancel{S} = \{(0,P),(0,N),(1,P),(1,N)\}$$

where 0 means person does not have disease

1 means does have disease

Consider two events:

$A = \{$person has the disease$\}$

$= \{(1,P),(1,N)\}$

$B = \{$ person tests positive $\}$

$= \{(0,P),(1,P)\}$

Q1 asks for $P(A \cap B)$, but for Q2, Q3, we need:

Defn: Given two events A and B, the <u>conditional probability of A given B</u> is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) \neq 0$. If $P(B) = 0$, then $P(A|B)$ is undefined in this class. It is your responsibility to make sure that $P(B) \neq 0$ if you write $P(A|B)$.