1/19/2023

The Probability Measure: The Axiomatic Approach

The Probability measure P in the prob. space $(S, \mathcal{F}, P)$ is a set function from $\mathcal{F}$ to $\mathbb{R}$, denoted $P: \mathcal{F} \to \mathbb{R}$

The axiomatic approach works by defining a set of properties, or <u>axioms</u>, that a function P must satisfy to be a valid prob. measure.

The Axioms of Probability

A prob. measure is a function $P: \mathcal{F} \to \mathbb{R}$ that satisfies the following axioms:

(1). $P(A) \geq 0 \quad \forall A \in \mathcal{F}$

for all, or for any

— a prob cannot be negative

~~The Axioms of Probability~~

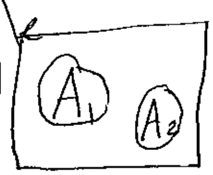
2. $P(S) = 1$

   - The prob. that one of the possible outcomes occurs is 1.

3. For any <u>disjoint</u> events $A_1, A_2 \in \mathcal{F}$,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

Example



   - $P(A_1$ or A_2 occurring$) = P(A_1) + P(A_2)$

   - For general A_1, A_2, $P(A_1 \cup A_2)$ is not necessarily $P(A_1) + P(A_2)$

4. For any <u>disjoint</u> sequence of events $A_1, A_2, \ldots$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

— extension of Axiom 3 to countably infinite case

Comments:

- There are other ways the axioms can be stated, but you are expected to use these four axioms in this class.
- Axiom 3 can be used to show that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

if $A_1, \ldots, A_n$ are disjoint, for finite $n \geq 2$

Proof by induction

- Axiom 4 implies Axiom 3, but if S is finite, ~~you~~

Axiom 4 follows from Axiom 3, Since if ~~Assume~~ S is finite, then there can only be a finite # of distinct events. In this case, let $A_1, A_2, \ldots$ be a sequence of disjoint events. Then there is some finite n for which

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} B_i$$

for some disjoint $B_1, \ldots, B_n$. So

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i)$$

by Axiom 3.

So if S is finite, you do not need to show Axiom 4!

(5)

- Since Axiom 4 implies Axiom 3 (proof omitted), in general you only need to show that Axioms 1, 2, 4 hold to show P is valid

Classical Approaches to Probability: The Counting Approach

- Consider a finite sample space S. Let A be an event in $\mathcal{F}$, and let

$$P(A) = \frac{|A|}{|S|} \quad \forall A \in \mathcal{F}$$

where $|A|$ = the number of elements in A

Example. Deal 5 cards. Let $S = \{(x_1, x_2, x_3, x_4, x_5):$ x_i is a card (value and suit)$\}$

then the event of getting four-of-a-kind is

$A = \{(x_1, x_2, x_3, x_4, x_5):$ 4 x_i's have the same value$\}$

then

$$P(A) = \frac{\text{\# of hands with 4-of-a-kind}}{\text{total \# of hands}}$$

under the counting approach.

- The counting approach is limited to cases where S is finite and where outcomes are equally likely

Classical Approaches to Probability: The Relative Frequency Approach

Also called the empirical approach; consider a sample space S and an event A. For some integer $n \geq 1$, let

$$P_n(A) = \frac{n_A}{n},$$

where n_A = # of times that A occurs in n trials of the experiment.

Can define $P(A)$ as $\lim_{n \to \infty} P_n(A)$.

— The relative frequency approach is used in engineering, but often it is too expensive to do very many trials

Luckily, it can be shown that the relative frequency and counting definitions of prob. both satisfy the axioms

Proofs omitted

The axiomatic approach to prob. is much more general and powerful than either of these two classical approaches by themselves.

~~Some useful properties of~~

Some Properties of P Derived from the Axioms

(1) $P(\emptyset) = 0 \quad \forall (S, \mathcal{F}, P)$

Proof. Can write

$\emptyset = \emptyset \cap S$, so

$\emptyset$ and S are disjoint

Then by Axiom 3,

$$P(\emptyset \cup S) = P(\emptyset) + P(S)$$

But $\emptyset \cup S = S$, so

$$P(S) = P(\emptyset) + P(S), \text{ so}$$

$$P(\emptyset) = 1 - 1 = 0.$$

Common mistake

If $P(A) = 0$, that does not mean

$$A = \emptyset$$