Set Theory for Probability (cont'd from Jan 12)

Three Types of Sets

- A ~~set~~ A is <u>finite</u> if it contains a finite number of elements.
  We can write such an A as
  $$A = \{x_1, \ldots, x_n\}$$ for some finite n, $x_i \in \mathcal{S}$
- A set A is <u>countably infinite</u> if its elements can be counted, or listed, in a way that includes all elements, and if it is not finite. Can write as
  $$A = \{x_1, x_2, \ldots\}, \quad x_i \in \mathcal{S}$$
- Set A is <u>uncountable</u> if it is not finite or countably infinite.
  In this class, the uncountable sets we will

encounter are $\mathbb{R}$ and intervals in $\mathbb{R}$.

Collections of Sets

- Finite collection of sets can be written as $A_1, \ldots, A_n$ for finite $n \geq 1$
- Countably infinite collections (or sequences) of sets can be written

$$A_1, A_2, \ldots$$

Defn. The <u>union of an indexed collection of sets</u> is defined as

$$\bigcup_{i \in I} A_i = \{\omega \in S : \omega \in A_i \text{ for at least one } i \in I\}$$

where $I = \{1, \ldots, n\}$ or $I = \{1, 2, \ldots\}$ for a finite or countably infinite collection, respectively. Write as

$\bigcup_{i=1}^{n} A_i$ or $\bigcup_{i=1}^{\infty} A_i$

Defn. The intersection of an indexed collection of sets is

$$\bigcap_{i\in I} A_i = \{w \in S : w \in A_i \text{ for every } i \in I\}$$

Write as $\bigcap_{i=1}^{n} A_i$ or $\bigcap_{i=1}^{\infty} A_i$

Example. Let $A_i = [i, i+1]^{CR}$ for $i = 0, 1, \ldots$

Then $\bigcup_{i=0}^{\infty} A_i = [0, \infty) \equiv \mathbb{R}^+$ ← non-negative real numbers

Also, $\bigcap_{i=0}^{\infty} A_i = \phi$

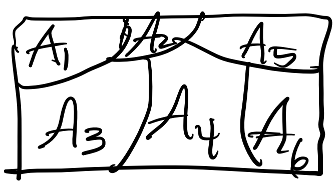
Defn. The collection of sets A_i, $i \in I$ is <u>disjoint</u> if

$$A_i \cap A_j = \phi \quad \forall i, j \in I, \; i \neq j$$

Defn. The collection A_i, $i \in I$, forms <u>a partition of S</u> if it is disjoint and if

$$\bigcup_{i\in I} A_i = S$$

Example:



The Probability Space

We use a "probability space" to model a random experiment mathematically. There are three components:

- The sample space S
- The event space $\mathcal{F}$
- The probability measure P

Can write as $(S, \mathcal{F}, P)$

The Sample Space $\mathcal{S}$

$\mathcal{S}$ is a non-empty set of elements called <u>outcomes</u>

Each time the experiment is run, exactly one outcome occurs.

Examples.

- A binary source generates a bit. Then $\mathcal{S} = \{0,1\}$ (finite $\mathcal{S}$)
- Count the number of particles hitting a detector. Then $\mathcal{S} = \{0,1,\dots\}$ (countably infinite $\mathcal{S}$)
- Measure the temperature in a room. Then $\mathcal{S} = [T_{min}, T_{max}] \subset \mathbb{R}$

interval

(uncountable S)

The Event Space

The event space is a collection of subsets of the sample space to which probabilities will be assigned

- Each event is a subset of S
- Each time the experiment is run, each event either occurs or does not occur
  - An event occurs if it contains the outcome that occurred.
- For the purposes of this class, you can assume any subset of S is a valid event
  - The math is more

complicated if $S = \mathbb{R}$, but you do not need to worry about that

- You do not need to worry about what the event space includes, since you can assume any subset of S we will encounter is an event

Examples. ① Let $S = \{H, T\}$

The we can let

$$\mathcal{F} = \{\phi, \{H\}, \{T\}, S\}$$

There are 4 events

② Let $S = \mathbb{R}$.

- $\mathcal{F}$ will contain all intervals (a, b) and $[a, b]$

- Also, $\{a\}$ is an event for any $a \in \mathbb{R}$.

Final point: Probabilities are assigned to events not outcomes.

Outcome or Event?

Example 1. $S = \{1, 2, 3\}$

(a) 1 outcome: $1 \in S$

(b) S event $S \subset S$

(c) $\{2, 3\} \subset S$, so event

(d) $\{2\} \subset S$ (elementary) event

(e) $\{2\} \cup \{3\}$ event (same event as in (c))

Example 2. $S = [0, 1] \subset \mathbb{R}$

(a) $(0, 1) \subset \mathbb{R}$ event

(b) $\frac{1}{2}$ outcome

(c) $\{[0,1)\}$

A set containing a subset of S is neither an outcome nor an event.

However if $S = \{$ intervals on $\mathbb{R}\}$, then $\{[0,1)\}$ is an (elementary) event