More math preliminaries:

Infinite series:

- Consider the example of tossing a fair coin until the first heads appears and then stop. What is the prob. of more than two tosses?

Consider TTH.

$$P(TTH) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3$$

assuming ind. tosses

Then

$$P(TTTH) = \left(\frac{1}{2}\right)^4. \text{ So}$$

$$P(>2 \text{ tosses}) = \boxed{\sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k}$$

Two basic results:

① $$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad -1<r<1$$

Geometric series

② $$\sum_{k=0}^{M} r^k = \frac{1-r^{M-1}}{1-r}, \quad -1<r<1, \; M>0$$

The Fundamental Theorem of Calculus states (essentially) that for a function f,

$$f(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt$$

Set Theory for Probability

Defn. A set is an unordered collection of objects, called

- elements, or
- members, or
- points

Notation: A set is denoted using $\{\ \}$

For example $\{1,2,3\}$ contains 3 elements: 1, 2, and 3

- This set could also be written as $\{3,1,2\}$, $\{3,2,1,1\}$, etc.
- If you write instead $(3,1,2)$, $(3,2,1)$, $(3,2,1,1)$, these are not sets

Some notation:

$\omega \in A$ ~ ω is an element of set A

$\omega \notin A$ ~ ω not in A

Two ways to specify a set

1. Explicitly list elements

   Examples:

   $$A = \{0, 1, 5\}$$

   $$A = \{H, T\}$$

   $$A = \{x_1, \dots, x_n\}$$

   $$A = \{x_1, x_2, \dots\}$$

2. Specify a rule for membership

   Example.

   $$A = \{\overset{\text{(all)}}{\omega} \in \underset{\text{integers}}{\mathbb{Z}} \underset{\text{such that}}{:} 1 \leq \omega \leq 6\}$$

Special case:

Intervals on $\mathbb{R}$. Examples:

$$A = (a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$A = [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Defn. Sets A and B are <u>equal</u>, written as $A = B$, if they contain exactly the same elements.

Defn. If $w \in A \implies w \in B$ for

implies that

sets A, B, then A is a <u>subset</u> of B. Write as $A \subset B$

Note that $A \subset B$ might mean $A = B$ in this class

Useful result for showing $A = B$ sometimes:

$A = B$ iff $A \subset B$ and $B \subset A$

Proof omitted

For an iff proof must show both if and only if parts

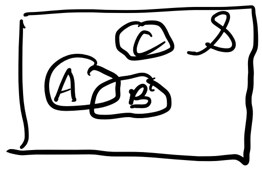
Defn. The set with no elements is called the <u>empty set</u>, or the <u>null set</u>, and denoted ϕ or $\{\}$

Common mistake: Writing $\{\phi\}$ insted of ϕ or $\{\}$

Defn. The set containing all possible elements of interest is call the universal set, or the space, denoted S

Venn Diagrams

A graphical representation of sets in a space



These can be useful for gaining insight for probability problems, but does not constitute a proof or an analytical solution to problems.

Set Operations

Given sets $A, B \subset S$:

Defn. The <u>intersection</u> of A and B is

$$A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$$

Defn. The <u>union</u> of A and B is

$$A \cup B = \{\omega \in S : \omega \in A \text{ or } \omega \in B \text{ or both}\}$$

Defn. The complement of A is

$$A^c = \{\omega \in S : \omega \notin A\}$$

Also denoted $\overline{A}$

Defn. If sets A and B have no elements in common, then they are disjoint. This can be written as $A \cap B = \emptyset$

Set Algebra

To show that two sets are equal, can sometimes use rules from set algebra. Some of these rules include:

- union and intersection are commutative and associative, so, e.g.,

$$A \cup B = B \cup A, \text{ and}$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- $\cap$ is distributive over $\cup$ (and vice versa), so

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- De Morgan's Laws:

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

Proofs left to the student