We are considering the distr. of a rv Y if $Y = g(X)$ for some rv X and some function $g: \mathbb{R} \to \mathbb{R}$. Continuing case ① where X, Y continuous, we have

$$F_Y(y) = P(g(X) \leq y) =$$
$$P(X \in A_y) = \int_{A_y} f_X(x)\,dx$$

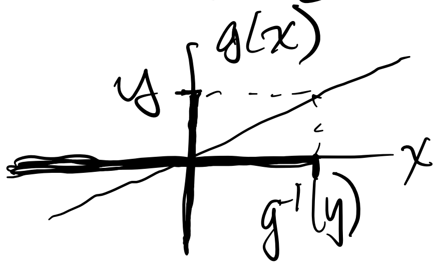
for some $A_y \subset \mathbb{R}$ that depends on y.

Note that if g is an invertible function (so for every $y \in \mathbb{R}$, there is a unique value x such that $g(x) = y$) then

$$\{Y \leq y\} = \{g(X) \leq y\}$$
$$= \{X \leq g^{-1}(y)\}$$

where $g^{-1}(y) = x$ if $y = g(x)$

Example. Let $Y = aX$ for some $a > 0$, $a \in \mathbb{R}$.



In this case $g(x) = ax$, so $g^{-1}(y) = \frac{y}{a}$

So $$F_Y(y) = P(aX \le y) = P\left(X \le \frac{y}{a}\right) =$$

$$\int_{-\infty}^{y/a} f_X(x)\,dx = F_X\left(\frac{y}{a}\right)$$

(Recall that $\int_{-\infty}^{x} f_X(t)\,dt = F_X(x)$)

Then the pdf of Y is

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right) \quad \forall y \in \mathbb{R}$$

Now consider an example where g is not invertible:

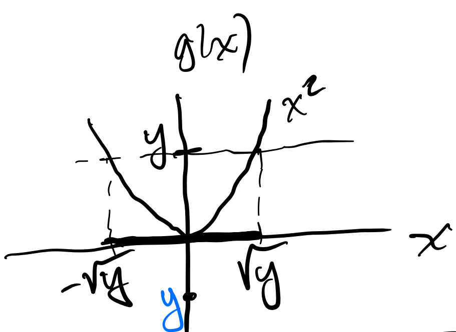
Example. Let $Y = X^2$

Common mistake:

$$P(Y \le y) = P(X^2 \le y) =$$

$$\cancel{P(X \le \sqrt{y}) = F_X(\sqrt{y})}$$

wrong



For $y > 0$, $A_y = [-\sqrt{y}, \sqrt{y}]$,

So
$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx$$

For $y < 0$, $P(Y \le y) = P(\phi) = 0$

since no value of X will lead to $Y = X^2 < 0$

For $y=0$: $P(Y=y) = P(X=0) = 0$

So $F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx & \text{if } y > 0 \end{cases}$

For $y>0$, we could also have written $F_Y(y) = P(-\sqrt{y} < X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

A second approach to finding f_Y when $Y=g(X)$, and X and Y are continuous is to use a change of variables:

Assume that g is an invertible function and that g is differentiable with $g'(x) \neq 0$. Then if can be shown that

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\left|\frac{dg(x)}{dx}\right|_{x=g^{-1}(y)}}$$

(Proof: Change of variables from Calculus)

The invertibility condition can be relaxed: if $g(x)$ has n solutions for a value y, or $g(x_1) = \cdots = g(x_n) = y$, then

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\left|\frac{dg(x)}{dx}\right|_{x=x_i}}$$

Using this approach for $Y = X^2$,

for $y > 0$, we have

$x_1 = -\sqrt{y}$, $x_2 = \sqrt{y}$

$$\frac{dg(x)}{dx} = 2x, \text{ so}$$

$$\left|\frac{dg(x)}{dx}\right|_{x=x_i} = 2x_i,$$

which gives

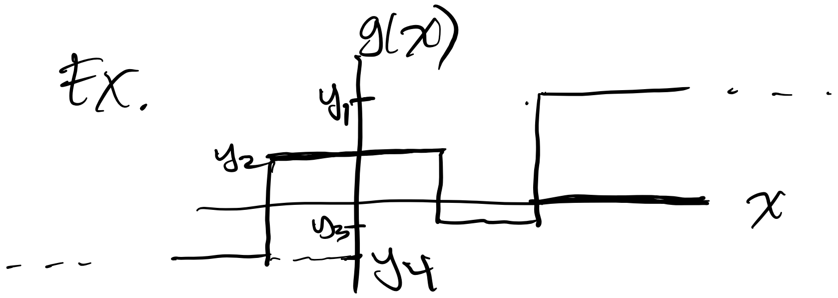
$$f_Y(y) = \frac{f_X(x_1)}{|2x_1|} + \frac{f_X(x_2)}{|2x_2|}$$

$$= \frac{f_X(-\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}}, \quad y > 0$$

Have finished case 1 for $Y = g(X)$, now consider

Case 2: X continuous, Y discrete.

Ex.



So for this example Y is discrete with

$$R_Y = \{y_1, y_2, y_3, y_4\}$$

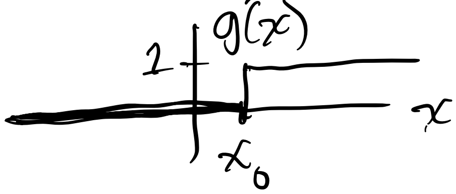
In general, it is best to first identify R_Y. Then for each $y \in R_Y$ find A_y such that

$$\{Y = y\} = \{g(X) = y\}$$
$$= \{X \in A_y\}, \quad A_y \subset \mathbb{R}$$

Then let $p_Y(y) = \int_{A_y} f_X(x)\,dx$

pmf of Y

Ex. Let $g(x) = u(x - x_0)$ for some $x_0 \in \mathbb{R}$ and let $Y = g(X)$. Find the pmf of Y.



We have $R_Y = \{0, 1\}$

Find $p_Y(0)$, $p_Y(1)$.

$$p_Y(0) = P(Y = 0) = P(X < x_0)$$
$$= \int_{-\infty}^{x_0} f_X(x)\,dx$$

Can use $p_Y(1) = 1 - p_Y(0)$

Since $\{Y = 1\} = \{Y = 0\}^c$