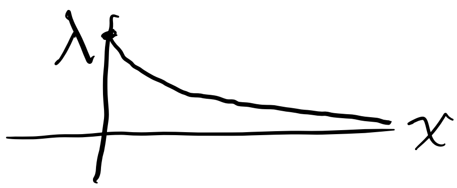
Some Common Random Variables (cont'd)

(5) Exponential

Continuous

$$f_X(x) = \lambda e^{-\lambda x} u(x), \forall x \in \mathbb{R}$$

$$\lambda \in \mathbb{R}, \lambda > 0$$



Used to model:

- lifetimes of certain devices or systems
- times between occurrences of certain events

(6) Poisson

Discrete

$$\mathcal{R}_X = \{0, \ldots, n\}$$

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, \ldots, n$$
$$\lambda \in \mathbb{R}, \lambda > 0$$

Used to model
- Some counting rvs, e.g., # of arrivals of customers, # of occurrences of certain events
- Noise in low SNR systems, e.g., limited- or low-photon imaging

There are other common rvs, but these are all we will cover here. You do not need to memorize the pdfs/pmfs given here.

Have now covered the

- cdf
- pdf
- pmf

as alternative representation of the measure $\mathbb{P}$ for rvs.

One common mistake:

Find the pdf of a rv X.

Response

wrong

$$f_X(x) = \begin{cases} \cdots & \text{if } X > x_0 \\ \cdots & \text{if } X \le x_0 \end{cases}$$

for some x_0

wrong

The pdf $f_X(x)$ is not random, but X is.

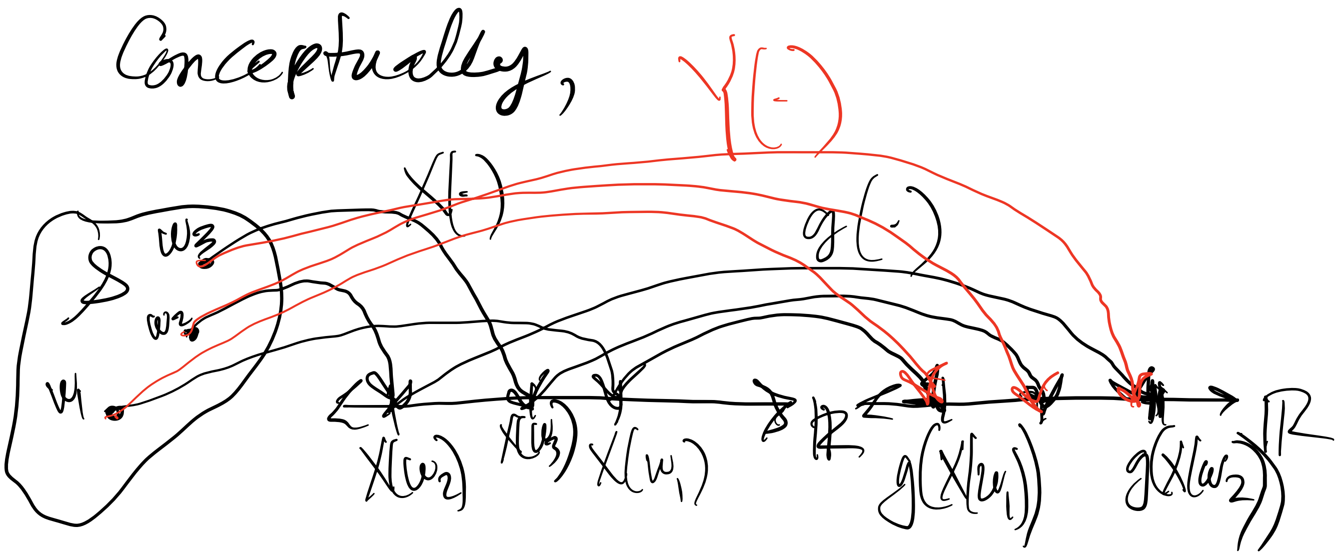
Functions of a Random Variable

Often we transform a rv using a function $g: \mathbb{R} \to \mathbb{R}$, to get a new rv. So $g(x)$ is a real number for each $x \in \mathbb{R}$, and is not random. However, if we consider

$Y = g(X)$, this means

$Y(w) = g(X(w))$, so Y is a rv since it depends on $w \in S$

Conceptually,

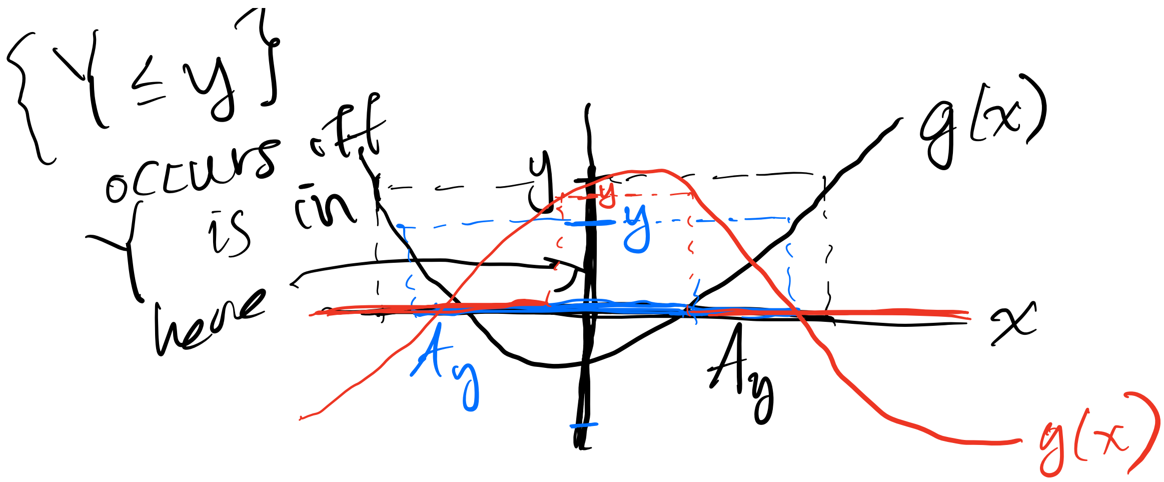


We want to find f_Y or p_Y from g and f_X or p_X.

We will consider 3 cases

1. X and Y both continuous

Key idea: Considering the x,y plane on which g can be graphed:



The event of interest is $\{Y \le y\}$ because we want to find $F_Y(y)$ and then let $f_Y(y) = \frac{d}{dy} F_Y(y)$.

We can find $F_Y(y)$ by finding an $A_y \subset \mathbb{R}$ such that

$$\{Y \le y\} = \{X \in A_y\}$$

A_y can be found from g, regardless of the distribution of X.

For each $y \in \mathbb{R}$, once A_y has been found, can let

$$F_Y(y) = \int_{A_y} f_X(x)\, dx$$

More formally, we write

$$F_Y(y) = P\left(Y \leq y\right) = P\left(g\left(X\right) \leq y\right)$$

$$= P\big(\underbrace{\{x \in \mathbb{R} : g\left(x\right) \leq y\}}\big) =$$

This event is $\{X \in A_y\}$ (see graph above)

$$P\left(X \in A_y\right)$$

Note: $\{Y \leq y\}$ occurs if and only if $g(X) \leq y$

because by definition

$$Y = g(X)$$

Also the event $\{g(X) \leq y\}$ occurs iff the event $\{X \in A_y\}$ occurs, because A_y was defined to make that true