Random Variables

Consider the example: For a fixed k, what is the prob. of k successes in n Bernoulli trials? Another way to frame this:

Let X be the number of successes in n Bernoulli trials. Then for $k = 0, 1, \ldots, n$,

$$\{X = k\} = B_k = \{k \text{ successes in } n \text{ Bernoulli trials}\}$$

To see that these events are equal for the case $n = 3$, $k = 2$ for the binary source example

- The event $\{X = 2\}$ contains outcomes $\{(0,1,0), (0,0,1), (1,0,0)\}$
- The event B_k contains exactly the same outcomes
- Since $\{X = k\} = B_k$,

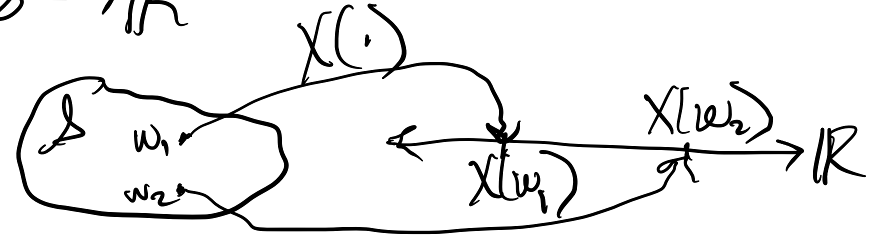
$$P(X=k) = P(B_k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Note that

- The value of X is random, since it cannot be found until the random experiment $(S, \mathcal{F}, P)$ has been run. Also if the rand. exp is run again, the value of X might change. So the number of successes in n Bernoulli trials is an example of a <u>random variable</u>

We will next give the general definition of a random variable, but this example illustrates the basic idea that a random variable is a variable whose value depends on the outcome of a random experiment.

Defn. Given a prob space $(S, \mathcal{F}, P)$, a <u>random variable</u> (rv) X is a function from S to the real numbers, $X: S \to \mathbb{R}$



Conceptually, the experiment $(S, \mathcal{F}, P)$ is run to get an outcome ω, which is then mapped to a value $X(\omega)$, usually written simply as X

Comments:

- In practice, $(S, \mathcal{F}, P)$ is not usually specified.
- For this class we will mostly consider two

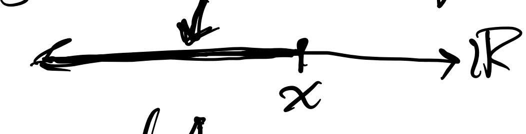
types of questions

① What is the prob of an event involving the rv X?

(For example, what is $P(X=k)$ in Bernoulli trials example.)

One important special case of an event involving rv X is $\{X \le x\}$ for $x \in \mathbb{R}$.

For a given x, the event $\{X \le x\}$ occurs if the value of $X(\omega)$ is in this region



Note: The variable x is not random. When x changes, the event

$\{X \le x\}$ changes, so as x changes, $P(X \le x)$ changes. We will soon define a function of x whose value is this prob.

(2) What value do we expect X to take?

This question will be addressed in later lectures

For the first question, we will define 3 functions that can be used to compute probs. of events related to rv X in practice:

- The cumulative

distribution function

- The probability density function
- the probability mass function (only for certain class of rvs)

The Distribution Function of X

Defn. The <u>cumulative distribution function (cdf)</u>, or simply distribution function of rv X is

$$F_X(x) = P(X \le x) \quad \forall x \in \mathbb{R}$$

can be changed to change $\{X \le x\}$

for all

subscript tells which rv cdf belongs to

Upper case F

Some properties of F_X:

Not every function $F: \mathbb{R} \to [0,1]$ is a valid cdf. This means for some

$F: \mathbb{R} \to [0,1]$, letting $P(X \le x) = F_X(x)$ would violate the axioms for some x. It can be shown that if the following 3 properties hold for a function F, then F is a valid cdf for some rv X:

① $\lim_{x \to -\infty} F(x) = 0$ and

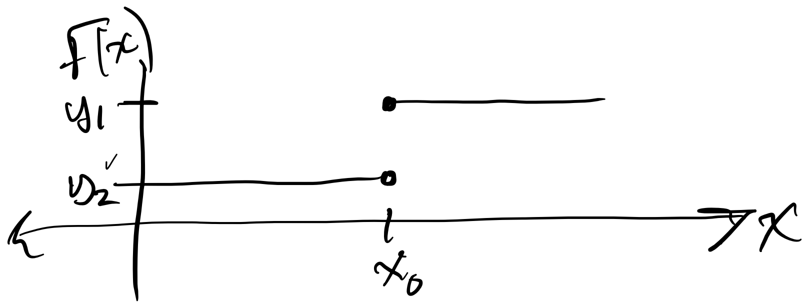
$\lim_{x \to \infty} F(x) = 1$

② For any $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$

$F(x_1) \le F(x_2)$, so

F is nondecreasing

③ F is continuous from the right at all $x \in \mathbb{R}$, i.e., $F(x^+) \equiv \lim_{\substack{\varepsilon \to 0, \\ \varepsilon > 0}} F(x+\varepsilon) = F_X(x)$ $\forall x \in \mathbb{R}$



$F_X(x_0)$ must be

y_1