Bernoulli Trials

Consider a random expt. $(S_0, \mathcal{F}_0, P_0)$, with one particular event A of interest. If we assume that n independent trials of the experiment are run, and we know $p = P_0(A)$, what is the prob. that A occurs k times in the n trials?

lower case

Ex. A binary source generates an independent sequence of n bits. What is the prob. that k of the bits are 0?

Combinatorics

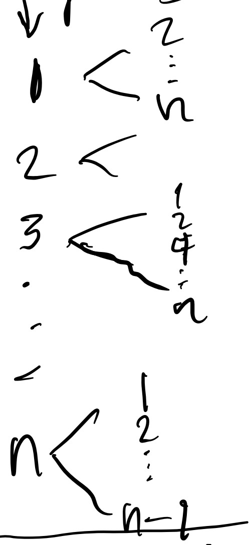
The question how many ways are there to choose k items from a total of n items needs to be answered to find the prob. of k successes in n trials for Bernoulli trials. This question is addressed by combinatorics

Consider the case of selecting k items from n total items with:

- no replacement
  - once picked, an item is not available to pick again, and
- order does not matter,

e.g., ab and ba count as one way.

Consider choosing k items from n. For the first choice, have n possibilities



$n \cdot (n-1)$ possibilities for first two choices

For each first choice, $n-1$ possible second choices, so the number of possibilities for the first two items is $n(n-1)$. This assumes that order matters. Each pair has been counted twice instead of once, so if order does not matter,

$$\frac{n(n-1)}{2} \text{ possibilities}$$

For general k, if order matters have

$$n(n-1)\cdots(n-k)(n-k+1)$$

If order <u>does not matter</u>, we have

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

because each combo has been counted $k!$ times

Normally we write this answer as

$$\frac{n(n-1)\cdots(n-k+1)\cancel{(n-k)\cdots(2)(1)}}{k!\,\cancel{(n-k)(n-k-1)\cdots(2)(1)}}$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k} \leftarrow \text{notation}$$

Call this value "n choose k"

Ex. How many 5-card hands have exactly 3 aces?

We need to consider two questions here:

How many ways can we choose 3 aces, and how many ways can we choose 2 "not aces"

There are $\binom{4}{3}$ combinations of 3 aces from the 4 aces in the deck

For each of these $\binom{4}{3}$ combos, there are $\binom{48}{2}$ ways to choose the 2 non-aces. So the # of 5-card hands with exactly 3 aces is $\binom{4}{3}\binom{48}{2} = 4512$

Why do we multiply n and n−1?

1	2, 3, 4	12, 13, 14
2	1, 3, 4	21, 23, 24
3	1, 2, 4	31, 32, 34
4	1, 2, 3	41, 42, 43

$n \times (n-1) = 12$

$n=4, k=2$ example:
How many ways to choose 2 from 4 items if order matters

n groups, n−1 items per group, so n(n−1) items

For the card example:

card 1 2 3 4 5

How many combos of 3 aces?

$$\begin{cases} A_1 \; A_2 \; A_3 \; 2_1 \; 3_2 \\ A_2 \; A_3 \; A_4 \\ \vdots \end{cases}$$

For each combo of three aces, how many combos of two non-aces?

Back to Bernoulli trials:

Let $S = \underbrace{S_0 \times \cdots \times S_0}_{n \text{ terms}}$

cartesian product

$= \{$n-tuples where each component is a value from $S_0\}$

This new sample space can be paired with an event space $\mathcal{F}$ that includes events of the form $\{k$ successes in n trials of experiment $(S_0, \mathcal{F}_0, P_0)\}$

We can let $\mathcal{F}$ be the collection of every subset of S

Example. For a sequence of 3 bits, the event of having exactly 2 of them be 0s is

$$\{(0,0,1),(0,1,0),(1,0,0)\}$$

end of example

Now we need P for our new prob. space $(S, \mathcal{F}, P)$. We will only define P here for events of the type

$B = \{k \text{ successes in } n \text{ trials}\}$ for $k = 0, 1, \ldots, n$

Will show that

$$\boxed{P(k \text{ successes in } n \text{ Bernoulli trials}) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Proof. There are $\binom{n}{k}$ ways (or elements in the event B). Each elementary event has prob. $p^k(1-p)^{n-k}$, using independence of the trials.

Then

$$P[B] = P\Big(\bigcup_{i=1}^{\binom{n}{k}} \underbrace{\{i\text{th combo}\}}_{\text{elementary event}}\Big)$$

Elementary events are disjoint

So $$P(B) = \sum_{i=1}^{\binom{n}{k}} \underbrace{p^k(1-p)^{n-k}}_{\text{does not depend on } i}$$,

So $$P[B] = \binom{n}{k} p^k (1-p)^{n-k}$$

Recall $p = P_0(A)$