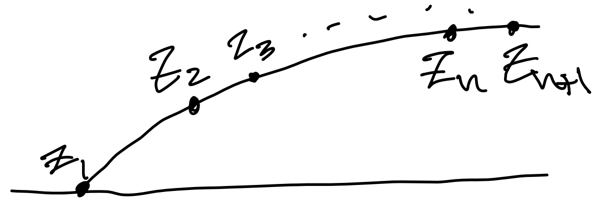
The Kalman Filter (KF)

This is an example application of discrete-time rps.

The KF estimates the state of a dynamic system from a sequence of noisy measurements of the state.



For this discussion, let the (true) state of the system be the distance of an object along this trajectory.

Goal: Estimate the distance z_n along the trajectory at time n (or t_n)

The Model: The KF uses a two-stage model

- The state model is

$$z_n = a_{n-1} z_{n-1} + w_{n-1}, \quad n = 1, 2, \ldots$$

with some initialization z_0,
where

$a_n \sim$ non-random sequence of state transition parameters

$w_n \sim$ state model error modeled as an additive white noise process:

$w_n \sim$ zero-mean

$E[w_n w_k] = 0$
$\forall n \neq k$

$E[w_n^2]$ is called the model error variance at time n

The KF tries to estimate z_n from a sequence of measurements

$x_1, \dots, x_n$

The observation (or measurement) model is

$X_n = Z_n + N_n$, where

N_n is noise in the measurement, modeled as additive white noise, so $E[N_n] = 0$,

$E[N_j N_k] = 0 \quad j \neq k$

$E[N_n^2]$ noise variance or power

One more assumption:

$\underline{E[N_j W_k] = 0} \quad \forall j, k$

Sensor noise and model error are uncorrelated

Estimation criterion:

The KF finds the linear minimum <u>mean-squared error</u> (MMSE) estimate of Z_n from $X_1, \ldots, X_{n-1}$.

Linear filter: Let

$$Y_n = \sum_{j=1}^{n} h_j^{(n-1)} X_{n-j} = h_1^{(n-1)} X_{n-1} + \cdots + h_n^{(n-1)} X_0$$

where $X_0 = Z_0$

Note: The KF is not time-invariant, so the values of $h_i^{(n-1)}$ depend on n

Y_n is our estimate of Z_n, so the MSE is

$$E\Big[\underbrace{(\underbrace{Y_n - Z_n}_{\text{error}})^2}_{\text{squared-error}}\Big]$$

mean

So we find the coefficients $h_1^{(n-1)}, \ldots, h_n^{(n-1)}$ that minimize

$$E\left[\left(\sum_{j=1}^{n} h_j^{(n-1)} X_{n-j} - Z_n\right)^2\right]$$

It can be shown that the

optimal filter satisfies

non-recursive

$$\boxed{R_{ZX}(n,l) = \sum_{j=1}^{n} h_j^{(n-1)} R_{XX}(n-j,l)}$$

$E[Z_n X_l]$ ~ cross-correlation function of rps Z_n and X_n

Need to solve n linear equations in n unknowns

Kalman developed a recursive version of this filter to make it practically realizable

Before presenting the recursive form, consider

- $R_{ZX}(n+1,l) = E[\underbrace{(a_n Z_n + W_n)}_{Z_{n+1} \text{ from state model}} X_l]$

$= a_n E[Z_n X_l] + E[W_n X_l],$

$l = 0, \ldots, n-1$

Model assumption:

$E[W_n X_l] = 0$, for

$l = 0, \ldots, n-1$

So $\underbrace{R_{ZX}(n+1, l) = a_n R_{ZX}(n, l)}_{\text{recursive}}$

- Also,

$R_{ZX}(n, l) = E[(\underbrace{X_n - N_n}_{Z_n \text{ from observation model}}) X_l]$

$= E[X_n X_l] - \underbrace{E[N_n X_l]}_{=0 \text{ for } l = 0, \ldots, n-1}$

So $R_{ZX}(n, l) = R_{XX}(n, l)$

So $R_{ZX}(n+1, l) = a_n R_{XX}(n, l)$

$l = 0, \ldots, n-1$